Discretization of the wave equation using continuous elements in time and a hybridizable discontinuous Galerkin method in space

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Abstract We provide an error analysis of two methods for time stepping the wave equation. These are based on the Hybridizable Discontinuous Galerkin (HDG) method to discretize in space, and the continuous Galerkin method to discretize in time. Two variants of HDG are proposed: a dissipative method based on the standard numerical flux used for elliptic problems, and a non-dissipative method based on a new choice of the flux involving time derivatives. The analysis of the fully discrete problem is based on simplified arguments using projections rather than explicit interpolants used in previous work. Some numerical results are shown that illuminate the theory.

Keywords discontinuous Galerkin method · hybridization · continuous time Galerkin method · error analysis · wave equation

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1 Introduction

The wave equation is often discretized using explicit time stepping methods. In this case the use of a discontinuous Galerkin method in space is attractive because these methods give a block diagonal mass matrix. However, if the grid is unstructured and more refined in some subregions of the domain compared to the average mesh size (for example to model small geometric features), an explicit method would require small time steps governed by the smallest elements to maintain stability (using a method that uses different time steps in different parts of the mesh is also a possible remedy in this case, see, e.g., Grote and Mitkova [8]).

As pointed out by Nguyen, Peraire and Cockburn in [16], discontinuous Galerkin methods are not generally well suited to implicit time stepping since the number of spatial degrees

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of freedom is generally larger for a discontinuous Galerkin method compared to a conforming method. However hybridizable discontinuous Galerkin (HDG) methods are well suited to implicit time stepping, since at each time step the element degrees of freedom can be eliminated from the problem by cheap local calculations so that only degrees of freedom associated with the numerical trace of the scalar variable on the skeleton of the mesh need to be involved with a global calculation at each time step. (A comparative study of the efficiency of conforming methods versus HDG schemes for elliptic problems has recently been carried out by Kirby, Sherwin and Cockburn in [14].)

In [16] interesting numerical results are presented for a scheme that combines the HDG method belonging to the general framework discussed by Cockburn, Gopalakrishnan and Lazarov [3] (for a refined analysis see Cockburn, Gopalakrishnan and Sayas [4] and for error estimates for the Helmholtz equation see [7]) together with backward differentiation formula or diagonally implicit Runge-Kutta (DIRK) methods in time. In particular it is shown that, provided the time stepping method is sufficiently high order, post-processing can be applied to extract inexpensive improved approximations to the fields. In addition energy conservation results are proved, but no error estimates are provided.

In this paper we shall prove convergence of a method that combines one of two variants of HDG methods in space with the continuous Galerkin method in time [5, 12, 13]. This time stepping method is closely related to the Gauss-Legendre Runge-Kutta method [9, 10]; in fact in our application the only difference resides in how the time integral of the source term is handled. We provide an analysis of the continuous time Galerkin method based on the work of [5, 13]. In particular we prove that the resulting methods are unconditionally stable, and analyze convergence.

We choose to analyze the continuous time Galerkin method in preference to a classical analysis of the Runge-Kutta scheme for several reasons. First it is noted in [1] that related Runge-Kutta and discontinuous Galerkin schemes can be viewed as approximations of the continuous time Galerkin method using suitable quadrature, so understanding the fundamental scheme is important. In addition, a variational analysis proves convergence under weaker conditions on the regularity of the solution than a classical analysis of the collocation scheme because the data is not interpolated or approximated by quadrature. Finally, Karakashian and Makridakis [12] note that the variational analysis is simpler than the analysis of corresponding implicit Runge-Kutta methods.

We rewrite the wave equation as a first order system involving velocity and pressure and use the HDG scheme to discretize in space. After discretization the resulting system of ordinary differential equations is similar to that arising from a mixed finite element method; (see Geveci [6] for a classical approach to mixed systems). We know of no convergence proof of the continuous time Galerkin method for the mixed system, but the approach we use could be applied in that case as well.

Of course the main drawback of such methods is that a linear system must be solved at each time step, and the size of the linear system increases with the number of stages (or order) of the time stepping scheme (this increase in size does not occur for the DIRK schemes in [16] and might be controlled by modifying the techniques from [17] previously used for parabolic problems, but this is not considered here).

Our analysis of the continuous time Galerkin method uses a decomposition of the discrete solution in time motivated by a similar decomposition used by French and Peterson in [5]. However, because that paper focused on a first order system in time but a second order system in space, we have to change the decomposition to enforce an orthogonality condition. This orthogonality condition is motivated by the analysis of continuous time Galerkin methods in [12, 13], where the discrete solution is written explicitly using basis functions

related to the Gauss-Legendre points in time. We avoid this explicit construction and work only with projections, and this simplifies the analysis considerably.

As mentioned earlier, we consider two variants of the HDG method. One is a straightforward generalization to the time domain of the method in [3], which is also the method used in [16]. As we shall see this method is dissipative, but also optimally convergent in space and time. By modifying the HDG fluxes to using time derivative terms, we can obtain a conservative scheme, but in that case do not prove optimal order convergence (missing by a factor of $h^{1/2}$ where h is the space grid size). Our limited numerical results do not exhibit this loss. In either case we use the projection proposed in [4] to provide an approximation operator in space.

In summary then, the novel content of this paper is to provide an analysis of the dissipative and new non-dissipative HDG methods in space using the continuous Galerkin method in time. We also provide a simplified analysis of the continuous time Galerkin method based on the work of [5,12,13]. Our analysis has relevance to other mixed systems.

The layout of the paper is as follows. In Section 2 we detail the problem we shall solve, and its discretization in space by the two HDG methods considered here. We derive energy relations for this semi-discrete problem and so show that one method is dissipative, while the new method is not. Then we detail the continuous time Galerkin - HDG (CTG-HDG) method. In Section 3 we introduce the decomposition of time dependent functions that underlies our analysis and provide an analysis of the decomposition, as well as several other approximation operators used in the analysis. Then in Section 4 we prove convergence of the dissipative CTG-HDG scheme by first deriving a stability result, and then verifying consistency before providing a final convergence result. The same general procedure is carried out for the non-dissipative scheme in Section 5. In Section 6 we provide a few numerical results to probe our theory. Finally in Section 7 we present some conclusions from our study.

2 The CTG-HDG method

2.1 Preliminaries

Given a final time T and a bounded Lipschitz polyhedral domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with boundary $\partial\Omega$ and functions $f = f(\boldsymbol{x}, t)$ and $g = g(\boldsymbol{x}, t)$, we consider the wave equation

$$\ddot{u} = \Delta u + f \qquad \text{in } \Omega \times (0, T) \tag{1a}$$

subject to the boundary data

$$u = g \qquad \text{on } \partial\Omega \times (0, T). \tag{1b}$$

Here $u(\boldsymbol{x}, t)$ describes the displacement in some direction of the point $\boldsymbol{x} \in \Omega$ at time $t \geq 0$ and we assume functions u_0 and u_1 are given such that

$$u(\cdot, 0) = u_0 \quad \text{and} \quad \dot{u}(\cdot, 0) = u_1 \qquad \text{in } \Omega. \tag{1c}$$

Throughout we denote by $\dot{w}$ and $\ddot{w}$ the first and second derivative of $w(\boldsymbol{x}, t)$ with respect to the variable t, respectively.

In this paper we wish to approximate the velocity $v = \dot{u}$ as well as the pressure $\boldsymbol{p} = \nabla u$. To apply the HDG method in space we rewrite (1) as a first order system

$$\dot{v} = \nabla \cdot \boldsymbol{p} + f \qquad\qquad \text{in } \Omega \times (0, T), \tag{2a}$$
$$\dot{\boldsymbol{p}} = \nabla v \qquad\qquad \text{in } \Omega \times (0, T), \tag{2b}$$

using boundary data

$$v = \dot{g} \qquad \text{on } \partial\Omega \times (0, T) \tag{2c}$$

and initial data

$$v(\cdot, 0) = v_0 := u_1 \quad \text{and} \quad \boldsymbol{p}(\cdot, 0) = \boldsymbol{p}_0 := \nabla u_0 \qquad \text{in } \Omega. \tag{2d}$$

As usual we consider a spatial mesh covering Ω consisting of regular, i.e., non-degenerate [2], tetrahedra K with faces F in $\mathbb{R}^3$ (or triangles K and edges F in $\mathbb{R}^2$) and denote the resulting collection of triangles/tetrahedra by $\mathcal{T}_h$. Accordingly h_K denotes the diameter of $K \in \mathcal{T}_h$ and $h := \max_K h_K$. The collection of edges/faces is denoted by $\mathcal{E}_h$, and the collection of element boundaries by $\partial\mathcal{T}_h := \{\partial K \mid K \in \mathcal{T}_h\}$.

On each element K and each edge/face F we consider the local spaces of polynomials of degree less than or equal to $k \geq 0$,

$$W(K) := \mathcal{P}_k(K), \quad \boldsymbol{V}(K) := \boldsymbol{\mathcal{P}}_k(K) := (\mathcal{P}_k(K))^d \quad \text{and} \quad M(F) := \mathcal{P}_k(F),$$

where $\mathcal{P}_k(S)$ denotes the space of polynomials of total degree at most k in m variables on S where $S \subset \mathbb{R}^m$. The corresponding global spatial finite element spaces are given by

$$W_h := \left\{ w \in L^2(\Omega) \mid w|_K \in W(K) \text{ for all } K \in \mathcal{T}_h \right\},$$
$$\boldsymbol{V}_h := \left\{ \boldsymbol{z} \in \boldsymbol{L}^2(\Omega) \mid \boldsymbol{z}|_K \in \boldsymbol{V}(K) \text{ for all } K \in \mathcal{T}_h \right\},$$
$$M_h := \left\{ \mu \in L^2(\mathcal{E}_h) \mid \mu|_F \in M(F) \text{ for all } F \in \mathcal{E}_h \right\},$$

where $\boldsymbol{L}^2(\Omega) := (L^2(\Omega))^d$ and $L^2(\mathcal{E}_h) := \prod_{F \in \mathcal{E}_h} L^2(F)$ are defined in the usual way. On these spaces we consider bilinear forms

$$(w, z)_{\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} (w, z)_K := \sum_{K \in \mathcal{T}_h} \int_K w \cdot z \, \mathrm{d}\boldsymbol{x},$$
$$(\boldsymbol{w}, \boldsymbol{z})_{\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} (\boldsymbol{w}, \boldsymbol{z})_K := \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{w} \cdot \boldsymbol{z} \, \mathrm{d}\boldsymbol{x},$$
$$\langle \eta, \mu \rangle_{\partial\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} \langle \eta, \mu \rangle_{\partial K} := \sum_{K \in \mathcal{T}_h} \int_{\partial K} \eta \mu \, \mathrm{d}s,$$

and write $\|\cdot\|_\Omega$, $\|\cdot\|_K$, $\|\cdot\|_{\partial\mathcal{T}_h}$ and $\|\cdot\|_{\partial K}$ for the corresponding L^2-norms.

We shall make use of the CGS projection originally introduced in [4], which is defined element by element as follows:

$$\Pi_h : L^2(\Omega) \times \boldsymbol{L}^2(\Omega) \supset \mathrm{dom}(\Pi_h) \to W_h \times \boldsymbol{V}_h, \quad \Pi_h(w, \boldsymbol{z}) := (\Pi_W w, \boldsymbol{\Pi_V} \boldsymbol{z}),$$

where for any $K \in \mathcal{T}_h$ and all edges/faces F of K the functions $\Pi_W w$ and $\boldsymbol{\Pi_V} \boldsymbol{z}$ satisfy

$$(\Pi_W w, \phi)_K = (w, \phi)_K \qquad \qquad \text{for all } \phi \in \mathcal{P}_{k-1}(K), \tag{3a}$$
$$(\boldsymbol{\Pi_V} \boldsymbol{z}, \boldsymbol{\psi})_K = (\boldsymbol{z}, \boldsymbol{\psi})_K \qquad \qquad \text{for all } \boldsymbol{\psi} \in \boldsymbol{\mathcal{P}}_{k-1}(K), \tag{3b}$$
$$\langle \boldsymbol{\Pi_V} \boldsymbol{z} \cdot \boldsymbol{n} - \tau \Pi_W w, \mu \rangle_F = \langle \boldsymbol{z} \cdot \boldsymbol{n} - \tau w, \mu \rangle_F \qquad \text{for all } \mu \in \mathcal{P}_k(F), \tag{3c}$$

where $\tau : \partial\mathcal{T}_h \to [0, \infty)$ denotes a non-negative function that is assumed to be constant on each edge/face of $K \in \mathcal{T}_h$ such that $\tau_K^{\max} := \max \tau|_{\partial K} > 0$ for all $K \in \mathcal{T}_h$, and $\boldsymbol{n}$ is the outward unit normal to K. The domain of definition $\mathrm{dom}\,\Pi_h$ of this projection is such that the right hand sides of (3) are well defined. Note that in (3c) we use a different sign than in the original

definition of the projection from [4]; this choice fits the structure of the numerical flux introduced in Section 2.2 below. The following approximation properties of the projection established in [4] remain true without changes: There exists a constant C independent of K and τ such that

$$\|\Pi_W w - w\|_K \leq C h_K^{\ell_w + 1} |w|_{H^{\ell_w + 1}(K)} + C \frac{h_K^{\ell_z + 1}}{\tau_K^{\max}} |\nabla \cdot z|_{H^{\ell_z}(K)}, \tag{4a}$$

$$\|\boldsymbol{\Pi_V} z - z\|_K \leq C h_K^{\ell_z + 1} |z|_{\boldsymbol{H}^{\ell_z + 1}(K)} + C h_K^{\ell_w + 1} \tau_K^* |w|_{H^{\ell_w + 1}(K)}, \tag{4b}$$

for $\ell_w, \ell_z \in [0, k]$, where $\tau_K^* := \max \tau|_{\partial K \setminus F^*}$ with F^* being an edge/face of K at which $\tau|_{\partial K}$ is maximum. Furthermore, as shown in Appendix A,

$$\|\tau(\Pi_W w - w)\|_{\partial K} \leq C h_K^{1/2} \left(h_K^{\ell_z} |\nabla \cdot z|_{H^{\ell_z}(K)} + \tau_K^{\max} h_K^{\ell_w} |w|_{H^{\ell_w + 1}(K)} \right) \tag{4c}$$

for $\ell_w, \ell_z \in [0, k]$. Here $H^s(S)$, $s \geq 0$, and $\boldsymbol{H}^s(S) := (H^s(S))^d$ are the standard Sobolev spaces on open domains $S \subset \mathbb{R}^d$, and we denote the corresponding semi-norms and norms by $|\cdot|_{H^s(S)}$, $|\cdot|_{\boldsymbol{H}^s(S)}$ and $\|\cdot\|_{H^s(S)}$, $\|\cdot\|_{\boldsymbol{H}^s(S)}$, respectively. Throughout C is a generic positive constant, not necessarily the same at different occurrences.

We shall need in addition the L^2-projection on the faces of the mesh $P_M : L^2(\partial \mathcal{T}_h) \to M_h$ defined by

$$\langle P_M \mu - \mu, \eta \rangle_{\partial \mathcal{T}_h} = 0 \qquad \text{for all } \eta \in M_h. \tag{5}$$

Of course, since M_h is a discontinuous space, this projection can be defined face by face.

2.2 Semi-discrete problem

The semi-discrete formulation of the method analyzed in this work consists in finding approximations $v_h(t) \in W_h$ of $v(t)$, $\boldsymbol{p}_h(t) \in \boldsymbol{V}_h$ of $\boldsymbol{p}(t)$ and a numerical trace $\widehat{v}_h(t) \in M_h$ approximating $v(t)$ on $\mathcal{E}_h$ for $t \in (0, T)$, which satisfy

$$(\dot{v}_h, \phi)_{\mathcal{T}_h} = -(\boldsymbol{p}_h, \nabla \phi)_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{p}}_h \cdot \boldsymbol{n}, \phi \rangle_{\partial \mathcal{T}_h} + (f, \phi)_{\mathcal{T}_h}, \tag{6a}$$

$$(\dot{\boldsymbol{p}}_h, \boldsymbol{\psi})_{\mathcal{T}_h} = -(v_h, \nabla \cdot \boldsymbol{\psi})_{\mathcal{T}_h} + \langle \widehat{v}_h, \boldsymbol{\psi} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h}, \tag{6b}$$

$$\langle \widehat{\boldsymbol{p}}_h \cdot \boldsymbol{n}, \mu \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} = 0, \tag{6c}$$

$$\langle \widehat{v}_h, \eta \rangle_{\partial \Omega} = \langle \dot{g}, \eta \rangle_{\partial \Omega}, \tag{6d}$$

for all $\phi \in W_h$, $\boldsymbol{\psi} \in \boldsymbol{V}_h$ and $\mu, \eta \in M_h$. Denoting by $\tau : \partial \mathcal{T}_h \to [0, \infty)$ a stabilization function that is constant on each edge/face, satisfying $\tau_K^{\max} > 0$ for all $K \in \mathcal{T}_h$ as before, we consider two possible methods corresponding to different numerical fluxes $\widehat{\boldsymbol{p}}_h$:

– A dissipative method closely related to the elliptic theory (see also [16]) where the flux is given by

$$\widehat{\boldsymbol{p}}_h = \boldsymbol{p}_h - \tau(v_h - \widehat{v}_h) \boldsymbol{n} \qquad \text{on } \partial \mathcal{T}_h. \tag{7}$$

– A new non-dissipative method that does not reduce to the elliptic problem at steady state where the flux is given by

$$\widehat{\boldsymbol{p}}_h = \boldsymbol{p}_h - \tau(\dot{v}_h - \dot{\widehat{v}}_h) \boldsymbol{n} \qquad \text{on } \partial \mathcal{T}_h. \tag{8}$$

We first show the energy conservation properties of the two methods. Selecting $\phi = v_h$ in (6a) and $\boldsymbol{\psi} = \boldsymbol{p}_h$ in (6b) we add the results and integrate by parts to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|v_h\|_\Omega^2 + \|\boldsymbol{p}_h\|_\Omega^2\big) = \langle\widehat{\boldsymbol{p}}_h \cdot \boldsymbol{n}, v_h\rangle_{\partial\mathcal{T}_h} + (f, v_h)_{\mathcal{T}_h} + \langle\widehat{v}_h - v_h, \boldsymbol{p}_h \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h}.$$

Combining (6c) with $\mu = \widehat{v}_h$ and (6d) with $\eta = \widehat{\boldsymbol{p}}_h \cdot \boldsymbol{n}$ together with this equality we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|v_h\|_\Omega^2 + \|\boldsymbol{p}_h\|_\Omega^2\big) = -\langle(\boldsymbol{p}_h - \widehat{\boldsymbol{p}}_h) \cdot \boldsymbol{n}, v_h - \widehat{v}_h\rangle_{\partial\mathcal{T}_h} + (f, v_h)_{\mathcal{T}_h}.$$

Proceeding first to the numerical flux (7) we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|v_h\|_\Omega^2 + \|\boldsymbol{p}_h\|_\Omega^2\big) = -\langle\tau(\widehat{v}_h - v_h), \widehat{v}_h - v_h\rangle_{\partial\mathcal{T}_h}, + (f, v_h)_{\mathcal{T}_h}.$$

i.e., if $f = 0$ this implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|v_h\|_\Omega^2 + \|\boldsymbol{p}_h\|_\Omega^2\big) \leq 0,$$

and the method is generally dissipative.

On the other hand, using the numerical flux in (8) we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|v_h\|_\Omega^2 + \|\boldsymbol{p}_h\|_\Omega^2 + \langle\tau(\widehat{v}_h - v_h), \widehat{v}_h - v_h\rangle_{\partial\mathcal{T}_h}\big) = (f, v_h)_{\mathcal{T}_h},$$

and the method is now non-dissipative provided the interface term is included in the energy.

2.3 Fully discrete problem

To discretize in time we consider a sequence of time steps

$$0 = t_0 < t_1 < t_2 < \cdots < t_N = T$$

where as before $T > 0$ is the final time for the integration. For $0 \leq n \leq N - 1$ we define $\Delta t_n := t_{n+1} - t_n$, $I_n := (t_n, t_{n+1})$, $\Delta t := \max_n \Delta t_n$, and for any $q \geq 1$ we consider the discrete spaces

$$\mathcal{S}_{q,\Delta t}^{(0)} := \{\varphi \in H^1((0,T)) \mid \varphi|_{I_n} \in \mathcal{P}_q(I_n),\, 0 \leq n \leq N - 1\},$$
$$\mathcal{S}_{q-1,\Delta t}^{(-1)} := \{\varphi \in L^2((0,T)) \mid \varphi|_{I_n} \in \mathcal{P}_{q-1}(I_n),\, 0 \leq n \leq N - 1\}.$$

Note that $\mathcal{S}_{q-1,\Delta t}^{(-1)}$ is a discontinuous space, whereas $\mathcal{S}_{q,\Delta t}^{(0)}$ is continuous (and one degree higher). The fully discrete space-time finite element spaces for the method are then

$$W_{h,\Delta t}^{(0)} := \mathcal{S}_{q,\Delta t}^{(0)} \otimes W_h, \qquad\qquad W_{h,\Delta t}^{(-1)} := \mathcal{S}_{q-1,\Delta t}^{(-1)} \otimes W_h,$$
$$\boldsymbol{V}_{h,\Delta t}^{(0)} := \mathcal{S}_{q,\Delta t}^{(0)} \otimes \boldsymbol{V}_h, \qquad\qquad \boldsymbol{V}_{h,\Delta t}^{(-1)} := \mathcal{S}_{q-1,\Delta t}^{(-1)} \otimes \boldsymbol{V}_h,$$
$$M_{h,\Delta t}^{(0)} := \mathcal{S}_{q,\Delta t}^{(0)} \otimes M_h, \qquad\qquad M_{h,\Delta t}^{(-1)} := \mathcal{S}_{q-1,\Delta t}^{(-1)} \otimes M_h.$$

Now we state the discrete problem. For later reference we formulate it in terms of more general right hand sides, which for the problem considered in this section are set to be

$\Phi := f$, $\boldsymbol{\Gamma} := 0$, $\Lambda := 0$ and $\Theta := \dot{g}$. We seek $(v_{h,\Delta t}, \boldsymbol{p}_{h,\Delta t}, \widehat{v}_{h,\Delta t}) \in W_{h,\Delta t}^{(0)} \times \boldsymbol{V}_{h,\Delta t}^{(0)} \times M_{h,\Delta t}^{(0)}$ such that

$$\int_0^T \left[(\dot{v}_{h,\Delta t}, \phi)_{\mathcal{T}_h} + (\boldsymbol{p}_{h,\Delta t}, \nabla \phi)_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{p}}_{h,\Delta t} \cdot \boldsymbol{n}, \phi \rangle_{\partial \mathcal{T}_h} \right] \mathrm{d}t \tag{9a}$$

$$= \int_0^T \left[(\Phi, \phi)_{\mathcal{T}_h} - \langle \Lambda, \phi \rangle_{\partial \mathcal{T}_h} \right] \mathrm{d}t,$$

$$\int_0^T \left[(\dot{\boldsymbol{p}}_{h,\Delta t}, \boldsymbol{\psi})_{\mathcal{T}_h} + (v_{h,\Delta t}, \nabla \cdot \boldsymbol{\psi})_{\mathcal{T}_h} - \langle \widehat{v}_{h,\Delta t}, \boldsymbol{\psi} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} \right] \mathrm{d}t = \int_0^T (\boldsymbol{\Gamma}, \boldsymbol{\psi})_{\mathcal{T}_h} \, \mathrm{d}t, \tag{9b}$$

$$\int_0^T \langle \widehat{\boldsymbol{p}}_{h,\Delta t} \cdot \boldsymbol{n}, \mu \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} \, \mathrm{d}t = \int_0^T \langle \Lambda, \mu \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} \, \mathrm{d}t, \tag{9c}$$

$$\int_0^T \langle \widehat{v}_{h,\Delta t}, \eta \rangle_{\partial \Omega} \, \mathrm{d}t = \int_0^T \langle \Theta, \eta \rangle_{\partial \Omega} \, \mathrm{d}t, \tag{9d}$$

for all $\phi \in W_{h,\Delta t}^{(-1)}$, $\boldsymbol{\psi} \in \boldsymbol{V}_{h,\Delta t}^{(-1)}$ and $\mu, \eta \in M_{h,\Delta t}^{(-1)}$, together with either the dissipative numerical flux

$$\widehat{\boldsymbol{p}}_{h,\Delta t} = \boldsymbol{p}_{h,\Delta t} - \tau (v_{h,\Delta t} - \widehat{v}_{h,\Delta t}) \boldsymbol{n} \qquad \text{on } \partial \mathcal{T}_h \tag{10}$$

corresponding to (7), or the non-dissipative numerical flux

$$\widehat{\boldsymbol{p}}_{h,\Delta t} = \boldsymbol{p}_{h,\Delta t} - \tau (\dot{v}_{h,\Delta t} - \dot{\widehat{v}}_{h,\Delta t}) \boldsymbol{n} \qquad \text{on } \partial \mathcal{T}_h \tag{11}$$

corresponding to (8). At the initial time we assume that

$$v_{h,\Delta t}(\cdot, 0) = \Pi_W v_0 \quad \text{and} \quad \boldsymbol{p}_{h,\Delta t}(\cdot, 0) = \boldsymbol{\Pi_V} \boldsymbol{p}_0,$$

and the initial condition for the interface variable is $\widehat{v}_{h,\Delta t}(\cdot, 0) = P_M v_0|_{\partial \mathcal{T}_h}$. Other more convenient choices of initial data could be allowed at the cost of some extra details and terms in the error estimate.

Note that by choosing the test functions to vanish except on a particular time interval (which is possible because they are discontinuous) we can solve step-by-step in time (although for $q > 1$ we have to solve a system of discrete problems). In any case, the fact that we have an HDG system in space implies that we can reduce the problem at each step to solving for the interface variables, i.e., for the scalar numerical trace $\widehat{v}_{h,\Delta t}$. This is an attraction of HDG for higher order CG methods in time.

3 Preliminary estimates for stability and consistency

The convergence analysis for the two numerical fluxes will be handled in separate sections. But some parts of the stability analysis are common to both arguments. This is the subject of the present section.

We start by defining an L^2-projection $P_{\mathcal{S}} : L^2(0, T) \to \mathcal{S}_{q-1, \Delta t}^{(-1)}$ for time dependent functions such that on each subinterval I_n in time and for all $\varphi \in L^2(0, T)$

$$\int_{t_n}^{t_{n+1}} (P_{\mathcal{S}} \varphi - \varphi) \vartheta \, \mathrm{d}t = 0 \qquad \text{for all } \vartheta \in \mathcal{P}_{q-1}(I_n). \tag{12}$$

This projection suffices to prove stability in the lowest order case when $q = 1$, but we need another decomposition in order to provide stability between time steps for higher order elements in time. Suppose $\gamma \in \mathcal{P}_q(I_n)$, then we write

$$\gamma = \gamma_0 + \frac{(t - t_n)}{\Delta t} \gamma_1, \tag{13}$$

where $\gamma_0 \in \mathcal{P}_q(I_n)$ and $\gamma_1 \in \mathcal{P}_{q-1}(I_n)$ are such that

$$\int_{t_n}^{t_{n+1}} \gamma_0 \vartheta \, dt = 0 \qquad \text{for all } \vartheta \in \mathcal{P}_{q-1}(I_n). \tag{14}$$

We have the following lemma guaranteeing the existence and stability of this decomposition.

Lemma 1 *The decomposition* (13) *is well defined and in addition there is a constant C independent of n, Δt and γ such that for each $0 \leq n \leq N - 1$,*

$$\|\gamma_1\|_{L^2(I_n)} \leq C \|\gamma\|_{L^2(I_n)} \tag{15}$$

for all $\gamma \in \mathcal{P}_q(I_n)$. The component γ_0 satisfies

$$\gamma_0(t_n) = \gamma(t_n) \quad \text{and} \quad \gamma_0^2(t_{n+1}) = \gamma^2(t_n),$$

and there is a constant C independent of n, Δt and γ such that

$$\|\gamma_0\|_{L^2(I_n)} \leq C \Delta t^{1/2} |\gamma(t_n)|. \tag{16}$$

Remark 1 A similar decomposition to (13) was used in [5] but in their case γ_0 was taken to be constant in time. This suffices for the first order derivative in time and second order derivative in space formulation used there, but results in terms we cannot analyze for the first order system considered here. Our choice avoids these terms and is motivated by the analysis of [11–13] where a similar decomposition is constructed explicitly using Lagrange basis functions at Gauss-Legendre points. Our goal is to simplify this analysis by avoiding this explicit construction.

Proof The decomposition (13) is well defined because, using the orthogonality requirement (14), $\gamma_1 \in \mathcal{P}_{q-1}(I_n)$ satisfies

$$\int_{t_n}^{t_{n+1}} \frac{(t - t_n)}{\Delta t} \gamma_1 \vartheta \, dt = \int_{t_n}^{t_{n+1}} \gamma \vartheta \, dt \qquad \text{for all } \vartheta \in \mathcal{P}_{q-1}(I_n).$$

The left hand side of this equation defines a symmetric, bounded and coercive bilinear form on $\mathcal{P}_{q-1}(I_n) \times \mathcal{P}_{q-1}(I_n)$ considered as subspaces of $L^2(I_n) \times L^2(I_n)$. So by the Lax-Milgram lemma γ_1 exists uniquely for each γ. Furthermore,

$$\left\| \sqrt{\frac{t - t_n}{\Delta t}} \gamma_1 \right\|_{L^2(I_n)} \leq C \|\gamma\|_{L^2(I_n)}.$$

Mapping to the reference interval $[0, 1]$, using the equivalence of norms on finite dimensional vector spaces, and then mapping back to the given time interval establishes (15).

The fact that $\gamma_0(t_n) = \gamma(t_n)$ is obvious, and selecting $\vartheta = \dot{\gamma}_0$ in (14) shows that

$$0 = \int_{t_n}^{t_{n+1}} \gamma_0 \dot{\gamma}_0 \, dt = \frac{1}{2} \left(\gamma_0^2(t_{n+1}) - \gamma_0^2(t_n) \right).$$

By mapping to the reference interval,

$$\|\gamma_0\|^2_{L^2(I_n)} = \Delta t \int_0^1 \gamma_0^2(t_n + \Delta t s)\, ds.$$

Furthermore, denoting by π_q the Legendre polynomial of degree q on $[0,1]$, then in view of the fact that Gauss-Legendre quadrature using q points is accurate for polynomials of degree $2q-1$, we see that the orthogonality requirement (14) is equivalent to the fact that γ_0 vanishes at the shifted Legendre points. So

$$\gamma_0(t_n + \Delta t s) = \frac{\gamma(t_n)}{\pi_q(0)} \pi_q(s),$$

and thus,

$$\|\gamma_0\|^2_{L^2(I_n)} = \Delta t \left(\frac{\gamma(t_n)}{\pi_q(0)}\right)^2 \int_0^1 |\pi_q(s)|^2\, ds = \Delta t \left(\frac{\gamma(t_n)}{\pi_q(0)}\right)^2 \frac{1}{2q+1}.$$

Since q is fixed, this gives (16). $\square$

Given $w_{h,\Delta t} \in W_{h,\Delta t}^{(0)}$ we use the decomposition (13) to write

$$w_{h,\Delta t}|_{I_n} = w_0^{(n)} + \frac{(t-t_n)}{\Delta t} w_1^{(n)} \qquad \text{for each } 0 \le n \le N-1,$$

where $w_0^{(n)} \in \mathcal{P}_q(I_n) \otimes W_h$ and $w_1^{(n)} \in \mathcal{P}_{q-1}(I_n) \otimes W_h$, and we note that corresponding decompositions of $z_{h,\Delta t} \in V_{h,\Delta t}^{(0)}$ and $\widehat{w}_{h,\Delta t} \in M_{h,\Delta t}^{(0)}$ are given analogously. In the following lemma we establish stability estimates for these decompositions. We write $S_n := \mathcal{T}_h \times I_n$, $\partial S_n := \partial \mathcal{T}_h \times I_n$ and denote by $\|\cdot\|_{S_n} := \|\cdot\|_{L^2(I_n;L^2(\Omega))}$, $\|\cdot\|_{S_n} := \|\cdot\|_{L^2(I_n;L^2(\Omega))}$ as well as $\|\cdot\|_{\partial S_n} := \|\cdot\|_{L^2(I_n;L^2(\partial \mathcal{T}_h))}$ the usual Bochner space norms. The result should be compared to Lemma 2.1 of [12] and Lemma 3.4 of [13], where related estimates are given.

Lemma 2 *Suppose $w_{h,\Delta t} \in W_{h,\Delta t}^{(0)}$, $z_{h,\Delta t} \in V_{h,\Delta t}^{(0)}$ and $\widehat{w}_{h,\Delta t} \in M_{h,\Delta t}^{(0)}$. Then there is a constant C independent of $w_{h,\Delta t}$, $z_{h,\Delta t}$, $\widehat{w}_{h,\Delta t}$, n, Δt and h such that for each $0 \le n \le N-1$,*

$$\|w_{h,\Delta t}\|^2_{S_n} \le C\Delta t \left(\int_{t_n}^{t_{n+1}} (\dot{w}_{h,\Delta t}, w_1^{(n)})_{\mathcal{T}_h}\, dt + \|w_{h,\Delta t}(\cdot, t_n)\|^2_{\Omega} \right), \tag{17a}$$

$$\|z_{h,\Delta t}\|^2_{S_n} \le C\Delta t \left(\int_{t_n}^{t_{n+1}} (\dot{z}_{h,\Delta t}, z_1^{(n)})_{\mathcal{T}_h}\, dt + \|z_{h,\Delta t}(\cdot, t_n)\|^2_{\Omega} \right), \tag{17b}$$

$$\|\tau^{1/2}(\widehat{w}_{h,\Delta t} - w_{h,\Delta t})\|^2_{\partial S_n} \le C\Delta t \left(\int_{t_n}^{t_{n+1}} \langle \tau(\dot{\widehat{w}}_{h,\Delta t} - \dot{w}_{h,\Delta t}), \widehat{w}_1 - w_1 \rangle_{\partial \mathcal{T}_h}\, dt \right.$$
$$\left. + \|\tau^{1/2}(\widehat{w}_{h,\Delta t} - w_{h,\Delta t})(\cdot, t_n)\|^2_{\partial \mathcal{T}_h} \right). \tag{17c}$$

Proof We only prove (17a) and note that (17b) and (17c) follow analogously. By the definition of $w_0^{(n)}$ and $w_1^{(n)}$ and integration by parts

$$\int_{t_n}^{t_{n+1}} (\dot{w}_{h,\Delta t}, w_1^{(n)})_{\mathcal{T}_h}\, dt$$
$$= \int_{t_n}^{t_{n+1}} \left[(\dot{w}_0^{(n)}, w_1^{(n)})_{\mathcal{T}_h} + \frac{1}{\Delta t} \left(\|w_1^{(n)}\|^2_{\Omega} + (t-t_n)(\dot{w}_1^{(n)}, w_1^{(n)})_{\mathcal{T}_h} \right) \right] dt \tag{18}$$
$$= \int_{t_n}^{t_{n+1}} (\dot{w}_0^{(n)}, w_1^{(n)})_{\mathcal{T}_h}\, dt + \frac{1}{2\Delta t} \|w_1^{(n)}\|^2_{S_n} + \frac{1}{2} \|w_1^{(n)}(\cdot, t_{n+1})\|^2_{\Omega}.$$

The Cauchy-Schwarz inequality, an inverse inequality in time and (16) give

$$\left| \int_{t_n}^{t_{n+1}} (\dot{w}_0^{(n)}, w_1^{(n)})_{\mathcal{T}_h} \, dt \right| \leq \|\dot{w}_0^{(n)}\|_{S_n} \|w_1^{(n)}\|_{S_n} \leq \frac{C}{\Delta t} \|w_0^{(n)}\|_{S_n} \|w_1^{(n)}\|_{S_n}$$

$$\leq \frac{C}{\Delta t^{1/2}} \|w_{h,\Delta t}(\cdot, t_n)\|_{\Omega} \|w_1^{(n)}\|_{S_n}.$$

Hence, applying the arithmetic geometric mean inequality,

$$\left| \int_{t_n}^{t_{n+1}} (\dot{w}_0^{(n)}, w_1^{(n)})_{\mathcal{T}_h} \, dt \right| \leq C \|w_{h,\Delta t}(\cdot, t_n)\|_{\Omega}^2 + \frac{1}{4\Delta t} \|w_1^{(n)}\|_{S_n}^2,$$

and using this inequality in (18) gives

$$\int_{t_n}^{t_{n+1}} (\dot{w}_{h,\Delta t}, w_1^{(n)})_{\mathcal{T}_h} \, dt \geq \frac{1}{4\Delta t} \|w_1^{(n)}\|_{S_n}^2 + \frac{1}{2} \|w_1^{(n)}(\cdot, t_{n+1})\|_{\Omega}^2 - C \|w_{h,\Delta t}(\cdot, t_n)\|_{\Omega}^2.$$

However direct estimation recalling the fact that $0 \leq (t - t_n)/\Delta t \leq 1$ on I_n together with (16) shows that

$$\|w_{h,\Delta t}\|_{S_n}^2 \leq C \big(\|w_0^{(n)}\|_{S_n}^2 + \|w_1^{(n)}\|_{S_n}^2 \big) \leq C \big(\Delta t \|w_{h,\Delta t}(\cdot, t_n)\|_{\Omega}^2 + \|w_1^{(n)}\|_{S_n}^2 \big).$$

Combining these inequalities yields (17a).   □

When proving consistency, we shall use a generalized interpolant $P_L : H^1((0,T)) \to \mathcal{S}_{q,\Delta t}^{(0)}$ related to interpolation at the Gauss-Lobatto points (see [13] where an explicit Gauss-Lobatto interpolant is used) defined on the interval $[t_n, t_{n+1}]$ by

$$(P_L \varphi)(t_n) = \varphi(t_n), \qquad (P_L \varphi)(t_{n+1}) = \varphi(t_{n+1}), \tag{19a}$$

$$\int_{t_n}^{t_{n+1}} (P_L \varphi - \varphi)(t) \vartheta(t) \, dt = 0 \qquad \text{for all } \vartheta \in \mathcal{P}_{q-2}(I_n). \tag{19b}$$

This is a standard generalization of the interpolant and has the following properties.

Lemma 3 *The operator P_L is well defined for $\varphi \in H^1(0,T)$ and if $\varphi \in H^{q+1}(I_n)$ then*

$$\|\varphi - P_L \varphi\|_{L^2(I_n)} \leq C \Delta t^{q+1} \|\varphi\|_{H^{q+1}(I_n)}. \tag{20}$$

In addition, for all $\vartheta \in \mathcal{P}_{q-1}(I_n)$

$$\int_{t_n}^{t_{n+1}} \frac{d}{dt}(P_L \varphi) \vartheta \, dt = \int_{t_n}^{t_{n+1}} \dot{\varphi} \vartheta \, dt. \tag{21}$$

Proof The existence of P_L is well-known and follows from unisolvence, and the error estimate can be proved using the Bramble-Hilbert lemma [2]. The useful property (21) can be shown as follows. If $\vartheta \in \mathcal{P}_{q-1}(I_n)$, using partial integration and the end point interpolation and orthogonality properties of the interpolant,

$$\int_{t_n}^{t_{n+1}} \frac{d}{dt}(P_L \varphi) \vartheta \, dt = (P_L \varphi \vartheta)|_{t_n}^{t_{n+1}} - \int_{t_n}^{t_{n+1}} (P_L \varphi) \dot{\vartheta} \, dt = \int_{t_n}^{t_{n+1}} \dot{\varphi} \vartheta \, dt.$$

□

4 Analysis of the dissipative scheme

In this section we analyze the CTG-HDG method (9) assuming that the numerical flux $\widehat{\boldsymbol{p}}_{h,\Delta t}$ is given by (10).

4.1 Stability analysis for the dissipative scheme

We now suppose that $F \in L^2((0,T); L^2(\Omega))$ and $\boldsymbol{G} \in L^2((0,T); \boldsymbol{L}^2(\Omega))$ and prove the following result which provides local stability estimates for the dissipative scheme:

Theorem 1 *Let* $(w_{h,\Delta t}, \boldsymbol{z}_{h,\Delta t}, \widehat{w}_{h,\Delta t}) \in W^{(0)}_{h,\Delta t} \times \boldsymbol{V}^{(0)}_{h,\Delta t} \times M^{(0)}_{h,\Delta t}$ *satisfy* (9)-(10) *with* $\Phi = F$, $\boldsymbol{\Gamma} = \boldsymbol{G}$, $\Lambda = 0$, *and* $\Theta = 0$ *for all* $\phi \in W^{(-1)}_{h,\Delta t}$, $\boldsymbol{\psi} \in \boldsymbol{V}^{(-1)}_{h,\Delta t}$ *and* $\mu, \eta \in M^{(-1)}_{h,\Delta t}$. *Then for each* $0 \le n \le N-1$,

$$\|w_{h,\Delta t}(\cdot, t_{n+1})\|^2_{\Omega} + \|\boldsymbol{z}_{h,\Delta t}(\cdot, t_{n+1})\|^2_{\Omega} \le \|w_{h,\Delta t}(\cdot, t_n)\|^2_{\Omega} + \|\boldsymbol{z}_{h,\Delta t}(\cdot, t_n)\|^2_{\Omega}$$
$$+ 2 \int_{t_n}^{t_{n+1}} \left[(F, P_{\mathcal{S}} w_{h,\Delta t})_{\mathcal{T}_h} + (\boldsymbol{G}, P_{\mathcal{S}} \boldsymbol{z}_{h,\Delta t})_{\mathcal{T}_h} \right] \, \mathrm{d}t, \quad (22\mathrm{a})$$

and assuming that Δt *is small enough, there is a constant C independent of n,* Δt *and h as well as F and* $\boldsymbol{G}$ *such that*

$$\|w_{h,\Delta t}\|^2_{S_n} + \|\boldsymbol{z}_{h,\Delta t}\|^2_{S_n} \le C\Delta t \left(\|w_{h,\Delta t}(\cdot, t_n)\|^2_{\Omega} + \|\boldsymbol{z}_{h,\Delta t}(\cdot, t_n)\|^2_{\Omega} + \|F\|^2_{S_n} + \|\boldsymbol{G}\|^2_{S_n} \right). \quad (22\mathrm{b})$$

Proof Since the test functions are discontinuous in time we can consider a single time subinterval I_n and first choose $\phi = P_{\mathcal{S}} w_{h,\Delta t}$ and $\boldsymbol{\psi} = P_{\mathcal{S}} \boldsymbol{z}_{h,\Delta t}$, where $P_{\mathcal{S}}$ is the L^2-projection from (12). Then adding (9a) and (9b) with $\Phi = F$, $\boldsymbol{\Gamma} = \boldsymbol{G}$ and $\Lambda = 0$ gives

$$\int_{t_n}^{t_{n+1}} \left[(\dot{w}_{h,\Delta t}, P_{\mathcal{S}} w_{h,\Delta t})_{\mathcal{T}_h} + (\dot{\boldsymbol{z}}_{h,\Delta t}, P_{\mathcal{S}} \boldsymbol{z}_{h,\Delta t})_{\mathcal{T}_h} \right] \, \mathrm{d}t$$
$$= \int_{t_n}^{t_{n+1}} \left[-(\boldsymbol{z}_{h,\Delta t}, \nabla P_{\mathcal{S}} w_{h,\Delta t})_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{z}}_{h,\Delta t} \cdot \boldsymbol{n}, P_{\mathcal{S}} w_{h,\Delta t} \rangle_{\partial \mathcal{T}_h} \right] \, \mathrm{d}t$$
$$+ \int_{t_n}^{t_{n+1}} \left[-(w_{h,\Delta t}, \nabla \cdot P_{\mathcal{S}} \boldsymbol{z}_{h,\Delta t})_{\mathcal{T}_h} + \langle \widehat{w}_{h,\Delta t}, P_{\mathcal{S}} \boldsymbol{z}_{h,\Delta t} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} \right] \, \mathrm{d}t$$
$$+ \int_{t_n}^{t_{n+1}} \left[(F, P_{\mathcal{S}} w_{h,\Delta t})_{\mathcal{T}_h} + (\boldsymbol{G}, P_{\mathcal{S}} \boldsymbol{z}_{h,\Delta t})_{\mathcal{T}_h} \right] \, \mathrm{d}t.$$

Using the fact that $P_{\mathcal{S}}$ is a projection, integrating by parts and applying the definition of the numerical flux (10) we have

$$\int_{t_n}^{t_{n+1}} \left[(\dot{w}_{h,\Delta t}, w_{h,\Delta t})_{\mathcal{T}_h} + (\dot{\boldsymbol{z}}_{h,\Delta t}, \boldsymbol{z}_{h,\Delta t})_{\mathcal{T}_h} \right] \, \mathrm{d}t = \int_{t_n}^{t_{n+1}} \langle \tau P_{\mathcal{S}} (\widehat{w}_{h,\Delta t} - w_{h,\Delta t}), P_{\mathcal{S}} w_{h,\Delta t} \rangle_{\partial \mathcal{T}_h} \, \mathrm{d}t$$
$$+ \int_{t_n}^{t_{n+1}} \langle P_{\mathcal{S}} \widehat{w}_{h,\Delta t}, P_{\mathcal{S}} \boldsymbol{z}_{h,\Delta t} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} \, \mathrm{d}t + \int_{t_n}^{t_{n+1}} \left[(F, P_{\mathcal{S}} w_{h,\Delta t})_{\mathcal{T}_h} + (\boldsymbol{G}, P_{\mathcal{S}} \boldsymbol{z}_{h,\Delta t})_{\mathcal{T}_h} \right] \, \mathrm{d}t. \quad (23)$$

Choosing $\mu = P_{\mathcal{S}} \widehat{w}_{h,\Delta t}$ in (9c) with $\Lambda = 0$, $\eta = P_{\mathcal{S}} \widehat{\boldsymbol{z}}_{h,\Delta t} \cdot \boldsymbol{n}$ in (9d) with $\Theta = 0$, using these identities together with the definition of the numerical flux (10) in (23) and integrating in

time shows that

$$\|w_{h,\Delta t}(\cdot, t_{n+1})\|_\Omega^2 + \|z_{h,\Delta t}(\cdot, t_{n+1})\|_\Omega^2$$
$$+ 2\int_{t_n}^{t_{n+1}} \langle \tau P_{\mathcal{S}}(\widehat{w}_{h,\Delta t} - w_{h,\Delta t}), P_{\mathcal{S}}(\widehat{w}_{h,\Delta t} - w_{h,\Delta t}) \rangle_{\partial \mathcal{T}_h} \, \mathrm{d}t$$
$$= \|w_{h,\Delta t}(\cdot, t_n)\|_\Omega^2 + \|z_{h,\Delta t}(\cdot, t_n)\|_\Omega^2 + 2\int_{t_n}^{t_{n+1}} \left[(F, P_{\mathcal{S}} w_{h,\Delta t})_{\mathcal{T}_h} + (\boldsymbol{G}, P_{\mathcal{S}} z_{h,\Delta t})_{\mathcal{T}_h} \right] \mathrm{d}t.$$

This implies (22a).

Next we choose $\phi = w_1^{(n)}$ and $\boldsymbol{\psi} = z_1^{(n)}$ in (9a) and (9b). Then adding these equations gives

$$\int_{t_n}^{t_{n+1}} \left[(\dot{w}_{h,\Delta t}, w_1^{(n)})_{\mathcal{T}_h} + (\dot{z}_{h,\Delta t}, z_1^{(n)})_{\mathcal{T}_h} \right] \mathrm{d}t$$
$$= \int_{t_n}^{t_{n+1}} \left[-(z_{h,\Delta t}, \nabla w_1^{(n)})_{\mathcal{T}_h} + \langle \widehat{z}_{h,\Delta t} \cdot \boldsymbol{n}, w_1^{(n)} \rangle_{\partial \mathcal{T}_h} \right] \mathrm{d}t \qquad (24)$$
$$+ \int_{t_n}^{t_{n+1}} \left[-(w_{h,\Delta t}, \nabla \cdot z_1^{(n)})_{\mathcal{T}_h} + \langle \widehat{w}_{h,\Delta t}, z_1^{(n)} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} \right] \mathrm{d}t$$
$$+ \int_{t_n}^{t_{n+1}} \left[(F, w_1^{(n)})_{\mathcal{T}_h} + (\boldsymbol{G}, z_1^{(n)})_{\mathcal{T}_h} \right] \mathrm{d}t.$$

Using the orthogonality property (14), letting $\xi^{(n)} := (t - t_n)/\Delta t$, and integrating by parts,

$$\int_{t_n}^{t_{n+1}} (z_{h,\Delta t}, \nabla w_1^{(n)})_{\mathcal{T}_h} \, \mathrm{d}t = \int_{t_n}^{t_{n+1}} (z_0^{(n)} + \xi^{(n)} z_1^{(n)}, \nabla w_1^{(n)})_{\mathcal{T}_h} \, \mathrm{d}t$$
$$= \int_{t_n}^{t_{n+1}} \left[\langle \xi^{(n)} z_1^{(n)} \cdot \boldsymbol{n}, w_1^{(n)} \rangle_{\partial \mathcal{T}_h} - (\xi^{(n)} \nabla \cdot z_1^{(n)}, w_1^{(n)})_{\mathcal{T}_h} \right] \mathrm{d}t.$$

Thus, applying again the orthogonality property (14) together with this identity in (24) gives

$$\int_{t_n}^{t_{n+1}} \left[(\dot{w}_{h,\Delta t}, w_1^{(n)})_{\mathcal{T}_h} + (\dot{z}_{h,\Delta t}, z_1^{(n)})_{\mathcal{T}_h} \right] \mathrm{d}t$$
$$= \int_{t_n}^{t_{n+1}} \left[\langle \xi^{(n)} \widehat{w}_1^{(n)}, z_1^{(n)} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} - \langle \xi^{(n)} (z_1^{(n)} - \widehat{z}_1^{(n)}) \cdot \boldsymbol{n}, w_1^{(n)} \rangle_{\partial \mathcal{T}_h} \right] \mathrm{d}t$$
$$+ \int_{t_n}^{t_{n+1}} \left[(F, w_1^{(n)})_{\mathcal{T}_h} + (\boldsymbol{G}, z_1^{(n)})_{\mathcal{T}_h} \right] \mathrm{d}t.$$

Proceeding as before using (9c) and (9d) together with (10) we obtain

$$\int_{t_n}^{t_{n+1}} \left[(\dot{w}_{h,\Delta t}, w_1^{(n)})_{\mathcal{T}_h} + (\dot{z}_{h,\Delta t}, z_1^{(n)})_{\mathcal{T}_h} \right] \mathrm{d}t + \int_{t_n}^{t_{n+1}} \langle \tau \xi^{(n)} (w_1^{(n)} - \widehat{w}_1^{(n)}), w_1^{(n)} - \widehat{w}_1^{(n)} \rangle_{\partial \mathcal{T}_h} \, \mathrm{d}t$$
$$= \int_{t_n}^{t_{n+1}} \left[(F, w_1^{(n)})_{\mathcal{T}_h} + (\boldsymbol{G}, z_1^{(n)})_{\mathcal{T}_h} \right] \mathrm{d}t. \qquad (25)$$

Now combining (17a), (17b) and (25) we have

$$\|w_{h,\Delta t}\|_{S_n}^2 + \|z_{h,\Delta t}\|_{S_n}^2 \le C\Delta t \left(\left| \int_{t_n}^{t_{n+1}} (F, w_1^{(n)})_{\mathcal{T}_h} \, \mathrm{d}t \right| \right.$$
$$\left. + \left| \int_{t_n}^{t_{n+1}} (\boldsymbol{G}, z_1^{(n)})_{\mathcal{T}_h} \, \mathrm{d}t \right| + \|w_{h,\Delta t}(\cdot, t_n)\|_\Omega^2 + \|z_{h,\Delta t}(\cdot, t_n)\|_\Omega^2 \right).$$

Using the Cauchy-Schwarz inequality, the arithmetic geometric mean inequality and (15) shows (22b), provided Δt is small enough. $\qquad \square$

Using this theorem we have the following corollary giving stability bounds on the solution of (9)-(10) with $\Phi = F$, $\boldsymbol{\Gamma} = \boldsymbol{G}$, $\Lambda = 0$, and $\Theta = 0$:

Corollary 1 *Let* $(w_{h,\Delta t}, \boldsymbol{z}_{h,\Delta t}, \widehat{w}_{h,\Delta t}) \in W_{h,\Delta t}^{(0)} \times \boldsymbol{V}_{h,\Delta t}^{(0)} \times M_{h,\Delta t}^{(0)}$ *satisfy* (9)-(10) *with* $\Phi = F$, $\boldsymbol{\Gamma} = \boldsymbol{G}$, $\Lambda = 0$, $\Theta = 0$ *and zero initial data for all* $\phi \in W_{h,\Delta t}^{(-1)}$, $\boldsymbol{\psi} \in \boldsymbol{V}_{h,\Delta t}^{(-1)}$ *and* $\mu, \eta \in M_{h,\Delta t}^{(-1)}$.

(i) Provided Δt *is small enough, the following estimates hold for each* $0 \le n \le N - 1$ *and* C *independent of* Δt, *n, h as well as* F *and* $\boldsymbol{G}$:

$$\|w_{h,\Delta t}(\cdot, t_{n+1})\|_\Omega^2 + \|\boldsymbol{z}_{h,\Delta t}(\cdot, t_{n+1})\|_\Omega^2 \le (Ct_{n+1} + 1) \sum_{m=0}^{n} \left(\|F\|_{S_m}^2 + \|\boldsymbol{G}\|_{S_m}^2 \right), \quad (26a)$$

$$\|w_{h,\Delta t}\|_{S_n}^2 + \|\boldsymbol{z}_{h,\Delta t}\|_{S_n}^2 \le C\Delta t (Ct_n + 1) \sum_{m=0}^{n} \left(\|F\|_{S_m}^2 + \|\boldsymbol{G}\|_{S_m}^2 \right). \quad (26b)$$

(ii) Assuming that Δt *is small enough and that* $\tau > 0$ *on* $\partial \mathcal{T}_h$, *the discrete problem* (9)-(10) *has a unique solution.*

Proof Assuming zero initial data we find from (22a) together with the algebraic arithmetic mean inequality and (22b) that for any $\delta > 0$

$$\|w_{h,\Delta t}(\cdot, t_{n+1})\|_\Omega^2 + \|\boldsymbol{z}_{h,\Delta t}(\cdot, t_{n+1})\|_\Omega^2$$

$$\le \sum_{m=0}^{n} \left[\frac{1}{\delta} \left(\|F\|_{S_m}^2 + \|\boldsymbol{G}\|_{S_m}^2 \right) + \delta \left(\|w_{h,\Delta t}\|_{S_m}^2 + \|\boldsymbol{z}_{h,\Delta t}\|_{S_m}^2 \right) \right]$$

$$\le \sum_{m=0}^{n} \left[\left(\frac{1}{\delta} + \delta C \Delta t \right) \left(\|F\|_{S_m}^2 + \|\boldsymbol{G}\|_{S_m}^2 \right) + \delta C \Delta t \left(\|w_{h,\Delta t}(\cdot, t_m)\|_\Omega^2 + \|\boldsymbol{z}_{h,\Delta t}(\cdot, t_m)\|_\Omega^2 \right) \right].$$

Denoting

$$\|w_{h,\Delta t}(\cdot, t_{m^*})\|_\Omega^2 + \|\boldsymbol{z}_{h,\Delta t}(\cdot, t_{m^*})\|_\Omega^2 := \max_{0 \le m \le n+1} \left(\|w_{h,\Delta t}(\cdot, t_m)\|_\Omega^2 + \|\boldsymbol{z}_{h,\Delta t}(\cdot, t_m)\|_\Omega^2 \right),$$

and choosing $\delta = 1/(2Ct_{m^*})$, we find that

$$\frac{1}{2} \left(\|w_{h,\Delta t}(\cdot, t_{m^*})\|_\Omega^2 + \|\boldsymbol{z}_{h,\Delta t}(\cdot, t_{m^*})\|_\Omega^2 \right) \le \sum_{m=0}^{m^*-1} \left(2Ct_{m^*} + \frac{1}{2} \right) \left(\|F\|_{S_m}^2 + \|\boldsymbol{G}\|_{S_m}^2 \right).$$

This implies (26a), and (26b) follows from (22b).

To prove unique solvability of (9)–(10) at each time step it suffices to show that $w_{h,\Delta t}$, $\boldsymbol{z}_{h,\Delta t}$ and $\widehat{w}_{h,\Delta t}$ are zero on I_n assuming that the right hand sides and the approximation of the solution at time step t_n, i.e., $w_{h,\Delta t}(\cdot, t_n)$, $\boldsymbol{z}_{h,\Delta t}(\cdot, t_n)$ and $\widehat{w}_{h,\Delta t}(\cdot, t_n)$ vanish (since the linear system is square). However, this follows immediately from (22), (25) and (17c). □

4.2 Consistency and convergence of the dissipative scheme

In this section we determine the consistency error for the dissipative scheme. To this end we study the projected errors $e_v := P_L \Pi_W v - v_{h,\Delta t}$ and $\boldsymbol{e_p} := P_L \boldsymbol{\Pi_V} \boldsymbol{p} - \boldsymbol{p}_{h,\Delta t}$, where P_L is the Lobatto interpolant from (19) and $(\Pi_W, \boldsymbol{\Pi_V})$ the projection introduced in (3). We also set $\widehat{e}_v := P_L P_M v - \widehat{v}_{h,\Delta t}$ on the faces of the mesh, applying the L^2-projection from (5). The following theorem shows that these discrete functions satisfy (9)-(10).

Theorem 2 *Setting* $w_{h,\Delta t} = e_v$, $z_{h,\Delta t} = \boldsymbol{e_p}$, $\widehat{w}_{h,\Delta t} = \widehat{e}_v$ *and*

$$\widehat{\boldsymbol{z}}_{h,\Delta t} = \widehat{\boldsymbol{e_p}} := \boldsymbol{e_p} - \tau(e_v - \widehat{e}_v)\boldsymbol{n} \qquad \text{on } \partial\mathcal{T}_h$$

the equations (9)-(10) *with* $\Phi = F$, $\boldsymbol{\Gamma} = \boldsymbol{G}$, $\Lambda = 0$, *and* $\Theta = 0$ *are satisfied for all* $\phi \in W_{h,\Delta t}^{(-1)}$, $\boldsymbol{\psi} \in \boldsymbol{V}_{h,\Delta t}^{(-1)}$ *and* $\mu, \eta \in M_{h,\Delta t}^{(-1)}$, *provided that*

$$F = \Pi_W \dot{v} - \dot{v} - (P_L \nabla \cdot \boldsymbol{p} - \nabla \cdot \boldsymbol{p}) \qquad \text{and} \qquad \boldsymbol{G} = \boldsymbol{\Pi_V} \dot{\boldsymbol{p}} - \dot{\boldsymbol{p}} - (P_L \nabla v - \nabla v).$$

Proof Starting with (9b) we can proceed time step by time step using the orthogonality properties of Π_W and $\boldsymbol{\Pi_V}$ stated in (3a)–(3b) and the fact that (9b) (with $\boldsymbol{\Gamma} = 0$) and (2b) are satisfied by the discrete and continuous solution, respectively, to show that for all $\boldsymbol{\psi} \in \boldsymbol{V}_{h,\Delta t}^{(-1)}$,

$$\int_{t_n}^{t_{n+1}} \left[(\boldsymbol{e_p}, \boldsymbol{\psi})_{\mathcal{T}_h} + (e_v, \nabla \cdot \boldsymbol{\psi})_{\mathcal{T}_h} - \langle \widehat{e}_v, \boldsymbol{\psi} \cdot \boldsymbol{n} \rangle_{\partial\mathcal{T}_h} \right] \, dt$$

$$= \int_{t_n}^{t_{n+1}} \left[\left(\frac{\partial}{\partial t} P_L \boldsymbol{\Pi_V} \boldsymbol{p}, \boldsymbol{\psi} \right)_{\mathcal{T}_h} + (P_L v, \nabla \cdot \boldsymbol{\psi})_{\mathcal{T}_h} - \langle \widehat{e}_v, \boldsymbol{\psi} \cdot \boldsymbol{n} \rangle_{\partial\mathcal{T}_h} \right] \, dt$$

$$- \int_{t_n}^{t_{n+1}} \langle \widehat{v}_{h,\Delta t}, \boldsymbol{\psi} \cdot \boldsymbol{n} \rangle_{\partial\mathcal{T}_h} \, dt - \int_{t_n}^{t_{n+1}} (\dot{\boldsymbol{p}} - \nabla v, \boldsymbol{\psi})_{\mathcal{T}_h} \, dt.$$

Now using the projection property of the Lobatto interpolant (21) and integration by parts we have

$$\int_{t_n}^{t_{n+1}} \left[(\dot{\boldsymbol{e_p}}, \boldsymbol{\psi})_{\mathcal{T}_h} + (e_v, \nabla \cdot \boldsymbol{\psi})_{\mathcal{T}_h} - \langle \widehat{e}_v, \boldsymbol{\psi} \cdot \boldsymbol{n} \rangle_{\partial\mathcal{T}_h} \right] \, dt$$

$$= \int_{t_n}^{t_{n+1}} \left[(\boldsymbol{\Pi_V} \dot{\boldsymbol{p}} - \dot{\boldsymbol{p}}, \boldsymbol{\psi})_{\mathcal{T}_h} - (P_L \nabla v - \nabla v, \boldsymbol{\psi})_{\mathcal{T}_h} \right] \, dt - \int_{t_n}^{t_{n+1}} \langle \widehat{e}_v - P_L v + \widehat{v}_{h,\Delta t}, \boldsymbol{\psi} \cdot \boldsymbol{n} \rangle_{\partial\mathcal{T}_h} \, dt.$$

But recalling the definition of $\widehat{e}_v$ and the projection P_M we have

$$\langle \widehat{e}_v - P_L v + \widehat{v}_{h,\Delta t}, \boldsymbol{\psi} \cdot \boldsymbol{n} \rangle_{\partial\mathcal{T}_h} = 0$$

and we conclude that (9b) is satisfied with $\boldsymbol{\Gamma} = \boldsymbol{G}$ as required.

Using the same arguments we can show that for all $\phi \in W_{h,\Delta t}^{(-1)}$,

$$\int_{t_n}^{t_{n+1}} \left[(\dot{e}_v, \phi)_{\mathcal{T}_h} + (\boldsymbol{e_p}, \nabla \phi)_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{e}}_v \cdot \boldsymbol{n}, \phi \rangle_{\partial\mathcal{T}_h} \right] \, dt \qquad (27)$$

$$= \int_{t_n}^{t_{n+1}} \left[(\Pi_W \dot{v} - \dot{v}, \phi)_{\mathcal{T}_h} - (P_L \nabla \cdot \boldsymbol{p} - \nabla \cdot \boldsymbol{p}, \phi)_{\mathcal{T}_h} \right] \, dt - \int_{t_n}^{t_{n+1}} \langle (\widehat{\boldsymbol{e}_p} + \widehat{\boldsymbol{p}}_{h,\Delta t} - P_L \boldsymbol{p}) \cdot \boldsymbol{n}, \phi \rangle_{\partial\mathcal{T}_h} \, dt.$$

Recalling the definition of $\widehat{\boldsymbol{e_p}}$,

$$\widehat{\boldsymbol{e_p}} \cdot \boldsymbol{n} = P_L \boldsymbol{\Pi_V} \boldsymbol{p} \cdot \boldsymbol{n} - \tau(P_L \Pi_W v - P_L P_M v) - \widehat{\boldsymbol{p}}_{h,\Delta t} \cdot \boldsymbol{n}. \qquad (28)$$

Therefore,

$$\int_{t_n}^{t_{n+1}} \langle (\widehat{\boldsymbol{e_p}} + \widehat{\boldsymbol{p}}_{h,\Delta t} - P_L \boldsymbol{p}) \cdot \boldsymbol{n}, \phi \rangle_{\partial\mathcal{T}_h} \, dt$$

$$= \int_{t_n}^{t_{n+1}} \langle (P_L \boldsymbol{\Pi_V} \boldsymbol{p} - P_L \boldsymbol{p}) \cdot \boldsymbol{n} - \tau(P_L \Pi_W v - P_L v), \phi \rangle_{\partial\mathcal{T}_h} \, dt = 0,$$

where we applied (3c). Using this in (27) shows that (9a) is satisfied provided $\Phi = F$ is chosen as in the theorem.

Next, (28), (3c) and (9c) (with $\Lambda = 0$) imply that for all $\mu \in M_{h,\Delta t}^{(-1)}$,

$$\int_{t_n}^{t_{n+1}} \langle \widehat{\boldsymbol{e}}_{\boldsymbol{p}} \cdot \boldsymbol{n}, \mu \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} \, \mathrm{d}t = \int_{t_n}^{t_{n+1}} \langle P_L(\boldsymbol{p} \cdot \boldsymbol{n} - \tau(v - v)), \mu \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} \, \mathrm{d}t = 0,$$

which shows that $\widehat{\boldsymbol{e}}_{\boldsymbol{p}}$ satisfies (9c) (with $\Lambda = 0$). Finally, for all $\eta \in M_{h,\Delta t}^{(-1)}$, we obtain from (1b) and (9d) (with $\Theta = \dot{g}$) that

$$\int_{t_n}^{t_{n+1}} \langle \widehat{e}_v, \eta \rangle_{\partial \Omega} \, \mathrm{d}t = \int_{t_n}^{t_{n+1}} \langle P_L(P_M v - v) + P_L v - \widehat{v}_{h,\Delta t}, \eta \rangle_{\partial \Omega} \, \mathrm{d}t = 0.$$

This shows that $\widehat{e}_v$ satisfies (9d) (with $\Theta = 0$) and completes the proof. $\square$

It will also be useful to have estimates of the consistency error, which follow immediately from (4a), (4b) and (20).

Lemma 4 *Assuming sufficient regularity on the solution (as indicated by the norms on the right hand side of the following inequalities), the consistency error terms in Theorem 2 are bounded as follows: For each $0 \le n \le N - 1$,*

$$\|F\|_{S_n} = \|(\Pi_W \dot{v} - \dot{v}) - (P_L \nabla \cdot \boldsymbol{p} - \nabla \cdot \boldsymbol{p})\|_{S_n}$$
$$\le C \Big[h^{\ell_v + 1} \|\dot{v}\|_{L^2(I_n; H^{\ell_v + 1}(\Omega))} + \frac{h^{\ell_{\boldsymbol{p}} + 1}}{\min_K \tau_K^{\max}} \|\nabla \cdot \dot{\boldsymbol{p}}\|_{L^2(I_n; H^{\ell_{\boldsymbol{p}} + 1}(\Omega))} + \Delta t_n^{q+1} \|\nabla \cdot \boldsymbol{p}\|_{H^{q+1}(I_n; L^2(\Omega))} \Big],$$

$$\|\boldsymbol{G}\|_{S_n} = \|(\boldsymbol{\Pi_V} \dot{\boldsymbol{p}} - \dot{\boldsymbol{p}}) - (P_L \nabla v - \nabla v)\|_{S_n}$$
$$\le C \Big[h^{\ell_{\boldsymbol{p}} + 1} \|\dot{\boldsymbol{p}}\|_{L^2(I_n; \boldsymbol{H}^{\ell_{\boldsymbol{p}} + 1}(\Omega))} + h^{\ell_v + 1} \max_K \tau v_K^* \|\dot{v}\|_{L^2(I_n; H^{\ell_v + 1}(\Omega))} + \Delta t_n^{q+1} \|\nabla v\|_{H^{q+1}(I_n; \boldsymbol{L}^2(\Omega))} \Big],$$

for $\ell_v, \ell_{\boldsymbol{p}} \in [0, k]$.

Using Corollary 1 and Theorem 2 together with Lemma 4 we have the following convergence result:

Theorem 3 *Suppose Δt is sufficiently small, and that the solution $(v, \boldsymbol{p})$ of the mixed hyperbolic problem (2) is sufficiently smooth in space and time that the estimates of Lemma 4 can be applied. In addition suppose $v_{h,\Delta t}$, $\boldsymbol{p}_{h,\Delta t}$ and $\widehat{v}_{h,\Delta t}$ satisfy the CTG-HDG problem (9) using the dissipative numerical flux (10) with $\tau = \mathcal{O}(1)$ and the initial conditions*

$$v_{h,\Delta t}(\cdot, 0) = \Pi_W v_0 \quad \text{and} \quad \boldsymbol{p}_{h,\Delta t}(\cdot, 0) = \boldsymbol{\Pi_V} \boldsymbol{p}_0 \qquad \text{in } \Omega$$

and $\widehat{v}_{h,\Delta t}(\cdot, 0) = P_M v_0|_{\partial \mathcal{T}_h}$ on $\partial \mathcal{T}_h$. Then the following error estimate holds:

$$\max_{0 \le n \le N} \big(\|v(\cdot, t_n) - v_{h,\Delta t}(\cdot, t_n)\|_\Omega + \|\boldsymbol{p}(\cdot, t_n) - \boldsymbol{p}_{h,\Delta t}(\cdot, t_n)\|_\Omega \big)$$
$$\le C T^{1/2} \big(h^{\ell_v + 1} M_1(T) + h^{\ell_{\boldsymbol{p}} + 1} M_2(T) + \Delta t^{q+1} M_3(T) \big),$$
$$\|v - v_{h,\Delta t}\|_{L^2([0,T]; L^2(\Omega))} + \|\boldsymbol{p} - \boldsymbol{p}_{h,\Delta t}\|_{L^2([0,T]; \boldsymbol{L}^2(\Omega))}$$
$$\le C T \big(h^{\ell_v + 1} M_1(T) + h^{\ell_{\boldsymbol{p}} + 1} M_2(T) + \Delta t^{q+1} M_3(T) \big),$$

for $\ell_v, \ell_{\boldsymbol{p}} \in [0, k]$, where

$$M_1(T) \le \|\dot{v}\|_{L^2([0,T]; H^{\ell_v + 1}(\Omega))},$$
$$M_2(T) \le \|\dot{\boldsymbol{p}}\|_{L^2([0,T]; \boldsymbol{H}^{\ell_{\boldsymbol{p}} + 1}(\Omega))} + \|\nabla \cdot \dot{\boldsymbol{p}}\|_{L^2([0,T]; H^{\ell_{\boldsymbol{p}} + 1}(\Omega))},$$
$$M_3(T) \le \|\nabla v\|_{H^{q+1}([0,T]; \boldsymbol{L}^2(\Omega))} + \|\nabla \cdot \boldsymbol{p}\|_{H^{q+1}([0,T]; L^2(\Omega))}.$$

5 Analysis of the non-dissipative scheme

This section contains the error analysis of the CTG-HDG scheme (9) together with the non-dissipative numerical flux from (11). Since the reasoning is similar to that in Section 4, we do not include all details and refer to that section instead whenever appropriate.

5.1 Stability of the non-dissipative scheme

Let $F \in L^2((0,T); L^2(\Omega))$, $\boldsymbol{G} \in L^2((0,T); \boldsymbol{L}^2(\Omega))$ and $H \in L^2((0,T); L^2(\partial \mathcal{T}_h))$. Similar to Theorem 1 we obtain the following local stability estimates for the non-dissipative scheme.

Theorem 4 *Suppose* $(w_{h,\Delta t}, \boldsymbol{z}_{h,\Delta t}, \widehat{w}_{h,\Delta t}) \in W_{h,\Delta t}^{(0)} \times \boldsymbol{V}_{h,\Delta t}^{(0)} \times M_{h,\Delta t}^{(0)}$ *satisfy* (9) *and* (11) *with* $\Phi = F$, $\boldsymbol{\Gamma} = \boldsymbol{G}$, $\Lambda = H$, *and* $\Theta = 0$ *for all* $\phi \in W_{h,\Delta t}^{(-1)}$, $\boldsymbol{\psi} \in \boldsymbol{V}_{h,\Delta t}^{(-1)}$ *and* $\mu, \eta \in M_{h,\Delta t}^{(-1)}$. *Then, for each* $0 \leq n \leq N-1$,

$$
\begin{aligned}
&\|w_{h,\Delta t}(\cdot, t_{n+1})\|_{\Omega}^2 + \|\boldsymbol{z}_{h,\Delta t}(\cdot, t_{n+1})\|_{\Omega}^2 + \|\tau^{1/2}(\widehat{w}_{h,\Delta t} - w_{h,\Delta t})(\cdot, t_{n+1})\|_{\partial \mathcal{T}_h}^2 \\
&\quad = \|w_{h,\Delta t}(\cdot, t_n)\|_{\Omega}^2 + \|\boldsymbol{z}_{h,\Delta t}(\cdot, t_n)\|_{\Omega}^2 + \|\tau^{1/2}(\widehat{w}_{h,\Delta t} - w_{h,\Delta t})(\cdot, t_n)\|_{\partial \mathcal{T}_h}^2 \\
&\quad + 2 \int_{t_n}^{t_{n+1}} \left[(F, P_{\mathcal{S}} w_{h,\Delta t})_{\mathcal{T}_h} + (\boldsymbol{G}, P_{\mathcal{S}} \boldsymbol{z}_{h,\Delta t})_{\mathcal{T}_h}\right] \mathrm{d}t + 2 \int_{t_n}^{t_{n+1}} \langle H, P_{\mathcal{S}}(\widehat{w}_{h,\Delta t} - w_{h,\Delta t})\rangle_{\partial \mathcal{T}_h} \mathrm{d}t,
\end{aligned}
\tag{29a}
$$

and assuming that Δt is small enough, there is a constant C independent of n, Δt and h as well as F, $\boldsymbol{G}$ and H such that

$$
\begin{aligned}
\|w_{h,\Delta t}\|_{S_n}^2 + \|\boldsymbol{z}_{h,\Delta t}\|_{S_n}^2 &+ \|\tau^{1/2}(\widehat{w}_{h,\Delta t} - w_{h,\Delta t})\|_{\partial S_n}^2 \\
&\leq C\Delta t \left(\|F\|_{S_n}^2 + \|\boldsymbol{G}\|_{S_n}^2 + \|\tau^{-1/2} H\|_{\partial S_n}^2 \right. \\
&\quad + \|w_{h,\Delta t}(\cdot, t_n)\|_{\Omega}^2 + \|\boldsymbol{z}_{h,\Delta t}(\cdot, t_n)\|_{\Omega}^2 + \|\tau^{1/2}(\widehat{w}_{h,\Delta t} - w_{h,\Delta t})(\cdot, t_n)\|_{\partial \mathcal{T}_h}^2).
\end{aligned}
\tag{29b}
$$

Proof As in the proof of Theorem 1 we consider a single time interval I_n and first choose test functions $\phi = P_{\mathcal{S}} w_{h,\Delta t}$, $\boldsymbol{\psi} = P_{\mathcal{S}} \boldsymbol{z}_{h,\Delta t}$, $\mu = P_{\mathcal{S}} \widehat{w}_{h,\Delta t}$ and $\eta = P_{\mathcal{S}} \widehat{\boldsymbol{z}}_{h,\Delta t} \cdot \boldsymbol{n}$. Adding (9a) and (9b) (with $\Phi = F$ and $\boldsymbol{\Gamma} = \boldsymbol{G}$ and $\Lambda = H$), integrating by parts, using (11), (9c) (with $\Lambda = H$) and (9d) (with $\Theta = 0$) we find that

$$
\begin{aligned}
\int_{t_n}^{t_{n+1}} &\left[(\dot{w}_{h,\Delta t}, w_{h,\Delta t})_{\mathcal{T}_h} + (\dot{\boldsymbol{z}}_{h,\Delta t}, \boldsymbol{z}_{h,\Delta t})_{\mathcal{T}_h}\right] \mathrm{d}t = -\int_{t_n}^{t_{n+1}} \langle \tau(\dot{\widehat{w}}_{h,\Delta t} - \dot{w}_{h,\Delta t}), \widehat{w}_{h,\Delta t} - w_{h,\Delta t}\rangle_{\partial \mathcal{T}_h} \mathrm{d}t \\
&+ \int_{t_n}^{t_{n+1}} \left[(F, P_{\mathcal{S}} w_{h,\Delta t})_{\mathcal{T}_h} + (\boldsymbol{G}, P_{\mathcal{S}} \boldsymbol{z}_{h,\Delta t})_{\mathcal{T}_h}\right] \mathrm{d}t + \int_{t_n}^{t_{n+1}} \langle H, P_{\mathcal{S}}(\widehat{w}_{h,\Delta t} - w_{h,\Delta t})\rangle_{\partial \mathcal{T}_h} \mathrm{d}t.
\end{aligned}
$$

Integrating in time gives (29a).

Next we choose $\phi = w_1^{(n)}$, $\boldsymbol{\psi} = \boldsymbol{z}_1^{(n)}$, $\mu = \widehat{w}_1^{(n)}$ and $\eta = \widehat{\boldsymbol{z}}_1^{(n)} \cdot \boldsymbol{n}$ and write $\xi^{(n)} := (t - t_n)/\Delta t$. Adding (9a) and (9b), integrating by parts, using the orthogonality property (14), recalling (11), (9c) and (9d) gives

$$
\begin{aligned}
\int_{t_n}^{t_{n+1}} &\left[(\dot{w}_{h,\Delta t}, w_1^{(n)})_{\mathcal{T}_h} + (\dot{\boldsymbol{z}}_{h,\Delta t}, \boldsymbol{z}_1^{(n)})_{\mathcal{T}_h}\right] \mathrm{d}t + \int_{t_n}^{t_{n+1}} \langle \tau(\dot{\widehat{w}}_{h,\Delta t} - \dot{w}_{h,\Delta t}), \widehat{w}_1^{(n)} - w_1^{(n)}\rangle_{\partial \mathcal{T}_h} \mathrm{d}t \\
&= \int_{t_n}^{t_{n+1}} \left[(F, w_1^{(n)})_{\mathcal{T}_h} + (\boldsymbol{G}, \boldsymbol{z}_1^{(n)})_{\mathcal{T}_h} + \langle H, \widehat{w}_1^{(n)} - w_1^{(n)}\rangle_{\partial \mathcal{T}_h}\right] \mathrm{d}t.
\end{aligned}
\tag{30}
$$

Combining (17) and (30) yields

$$\|w_{h,\Delta t}\|_{S_n}^2 + \|z_{h,\Delta t}\|_{S_n}^2 + \|\tau^{1/2}(\widehat{w}_{h,\Delta t} - w_{h,\Delta t})\|_{\partial S_n}^2$$

$$\leq C\Delta t\left(\left|\int_{t_n}^{t_{n+1}} (F, w_1^{(n)})_{\mathcal{T}_h} \, dt\right| + \left|\int_{t_n}^{t_{n+1}} (\boldsymbol{G}, \boldsymbol{z}_1^{(n)})_{\mathcal{T}_h} \, dt\right| + \left|\int_{t_n}^{t_{n+1}} \langle H, \widehat{w}_1^{(n)} - w_1^{(n)}\rangle_{\partial \mathcal{T}_h} \, dt\right|\right.$$

$$\left. + \|w_{h,\Delta t}(\cdot, t_n)\|_{\Omega}^2 + \|z_{h,\Delta t}(\cdot, t_n)\|_{\Omega}^2 + \|\tau^{1/2}(\widehat{w}_{h,\Delta t} - w_{h,\Delta t})(\cdot, t_n)\|_{\partial \mathcal{T}_h}^2\right).$$

Using the Cauchy-Schwarz inequality, the arithmetic geometric mean inequality and (15) shows that (29b) holds, provided Δt is small enough. $\qquad\square$

As in Section 4.1, this theorem immediately yields stability bounds on the solution of (9) and (11) with $\Phi = F$, $\boldsymbol{\Gamma} = \boldsymbol{G}$, $\Lambda = H$, and $\Theta = 0$:

Corollary 2 *Suppose* $(w_{h,\Delta t}, \boldsymbol{z}_{h,\Delta t}, \widehat{w}_{h,\Delta t}) \in W_{h,\Delta t}^{(0)} \times \boldsymbol{V}_{h,\Delta t}^{(0)} \times M_{h,\Delta t}^{(0)}$ *satisfy* (9) *and* (11) *with* $\Phi = F$, $\boldsymbol{\Gamma} = \boldsymbol{G}$, $\Lambda = H$, $\Theta = 0$ *and zero initial data for all* $\phi \in W_{h,\Delta t}^{(-1)}$, $\boldsymbol{\psi} \in \boldsymbol{V}_{h,\Delta t}^{(-1)}$ *and* $\mu, \eta \in M_{h,\Delta t}^{(-1)}$.

(i) Provided that Δt is small enough, the following estimates hold for each $0 \leq n \leq N-1$ and C independent of Δt, n and h as well as F, $\boldsymbol{G}$ and H:

$$\|w_{h,\Delta t}(\cdot, t_{n+1})\|_{\Omega}^2 + \|z_{h,\Delta t}(\cdot, t_{n+1})\|_{\Omega}^2 + \|\tau^{1/2}(\widehat{w}_{h,\Delta t} - w_{h,\Delta t})(\cdot, t_{n+1})\|_{\partial \mathcal{T}_h}^2 \quad (31\text{a})$$

$$\leq (Ct_{n+1} + 1)\sum_{m=0}^{n} \left(\|F\|_{S_m}^2 + \|\boldsymbol{G}\|_{S_m}^2 + \|\tau^{-1/2}H\|_{\partial S_m}^2\right),$$

$$\|w_{h,\Delta t}\|_{S_n}^2 + \|z_{h,\Delta t}\|_{S_n}^2 + \|\tau^{1/2}(\widehat{w}_{h,\Delta t} - w_{h,\Delta t})\|_{\partial S_n}^2 \quad (31\text{b})$$

$$\leq C\Delta t(Ct_n + 1)\sum_{m=0}^{n} \left(\|F\|_{S_m}^2 + \|\boldsymbol{G}\|_{S_m}^2 + \|\tau^{-1/2}H\|_{\partial S_m}^2\right).$$

(ii) Assuming that Δt is small enough and that $\tau > 0$ on $\partial \mathcal{T}_h$, the discrete problem (9) *together with* (11) *has a unique solution.*

Proof The proof of part (i) is analogous to that of part (i) of Corollary 1 and is therefore omitted. Unique solvability of (9) together with (11) at each time step follows from (29). $\quad\square$

5.2 Consistency and convergence of the non-dissipative scheme

As in Section 4.2 we consider the projected errors $e_v := P_L\Pi_W v - v_{h,\Delta t}$, $\boldsymbol{e_p} := P_L\boldsymbol{\Pi_V}\boldsymbol{p} - \boldsymbol{p}_{h,\Delta t}$ and $\widehat{e}_v := P_L P_M v - \widehat{v}_{h,\Delta t}$.

Theorem 5 *Setting* $w_{h,\Delta t} = e_v$, $\boldsymbol{z}_{h,\Delta t} = \boldsymbol{e_p}$, $\widehat{w}_{h,\Delta t} = \widehat{e}_v$ *and*

$$\widehat{\boldsymbol{z}}_{h,\Delta t} = \widehat{\boldsymbol{e}}_{\boldsymbol{p}} := \boldsymbol{e_p} - \tau(\dot{e}_v - \dot{\widehat{e}}_v)\boldsymbol{n} \qquad \text{on } \partial \mathcal{T}_h$$

the equations (9) *and* (11) *with* $\Phi = F$, $\boldsymbol{\Gamma} = \boldsymbol{G}$, $\Lambda = H$, *and* $\Theta = 0$ *are satisfied for all* $\phi \in W_{h,\Delta t}^{(-1)}$, $\boldsymbol{\psi} \in \boldsymbol{V}_{h,\Delta t}^{(-1)}$ *and* $\mu, \eta \in M_{h,\Delta t}^{(-1)}$, *provided that*

$$F = \Pi_W\dot{v} - \dot{v} - (P_L\nabla\cdot\boldsymbol{p} - \nabla\cdot\boldsymbol{p}),$$

$$\boldsymbol{G} = \boldsymbol{\Pi_V}\dot{\boldsymbol{p}} - \dot{\boldsymbol{p}} - (P_L\nabla v - \nabla v),$$

$$H = \tau P_L(\Pi_W v - v) - \tau(\Pi_W\dot{v} - \dot{v}).$$

Proof Proceeding time step by time step it can be seen as in the proof of Theorem 2 that (9b) is satisfied with $\boldsymbol{\Gamma} = \boldsymbol{G}$ as required, and that (27) holds for all $\phi \in W_{h,\Delta t}^{(-1)}$. Since

$$\widehat{\boldsymbol{e}}_{\boldsymbol{p}} \cdot \boldsymbol{n} = P_L \boldsymbol{\Pi_V} \boldsymbol{p} \cdot \boldsymbol{n} - \tau \frac{\partial}{\partial t} (P_L \Pi_W v - P_L P_M v) - \widehat{\boldsymbol{p}}_{h,\Delta t} \cdot \boldsymbol{n}, \tag{32}$$

we find using (21) and (3c) that

$$\int_{t_n}^{t_{n+1}} \langle (\widehat{\boldsymbol{e}}_{\boldsymbol{p}} + \widehat{\boldsymbol{p}}_{h,\Delta t} - P_L \boldsymbol{p}) \cdot \boldsymbol{n}, \phi \rangle_{\partial \mathcal{T}_h} \, \mathrm{d}t$$
$$= \int_{t_n}^{t_{n+1}} \left[\langle \tau P_L (\Pi_W v - v), \phi \rangle_{\partial \mathcal{T}_h} - \langle \tau (\Pi_W \dot{v} - \dot{v}), \phi \rangle_{\partial \mathcal{T}_h} \right] \, \mathrm{d}t.$$

Combining this with (27) shows that (9a) is satisfied provided $\Phi = F$ and $\Lambda = H$ are chosen as in the theorem.

Using (32), (3c), (21) and (9c) (with $\Lambda = 0$), it follows that for all $\mu \in M_{h,\Delta t}^{(-1)}$

$$\int_{t_n}^{t_{n+1}} \langle \widehat{\boldsymbol{e}}_{\boldsymbol{p}} \cdot \boldsymbol{n}, \mu \rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} \, \mathrm{d}t = \int_{t_n}^{t_{n+1}} \left[\langle \tau P_L (\Pi_W v - v), \mu \rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} - \langle \tau (\Pi_W \dot{v} - \dot{v}), \mu \rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} \right] \, \mathrm{d}t,$$

which yields (9c), and (9d) follows by the same argument used at the end of the proof of Theorem 2.   □

In addition to the bounds of Lemma 4 we have the following estimate for the consistency error on the mesh skeleton, which follows from (4c) and the continuity of P_L.

Lemma 5 *Assuming sufficient regularity on the solution (as indicated by the norms on the right hand side of the following inequality), the consistency error term on the mesh skeleton in Theorem 5 is bounded as follows: For each $0 \leq n \leq N-1$,*

$$\|H\|_{\partial S_n} = \|\tau P_L (\Pi_W v - v) - \tau (\Pi_W \dot{v} - \dot{v})\|_{\partial S_n}$$
$$\leq Ch^{1/2} \left[h^{\ell_{\boldsymbol{p}}} \|\nabla \cdot \boldsymbol{p}\|_{H^1(I_n; H^{\ell_{\boldsymbol{p}}}(\Omega))} + \max_K \tau_K^{\max} h^{\ell_v} \|v\|_{H^1(I_n; H^{\ell_v+1}(\Omega))} \right]$$

for $\ell_v, \ell_{\boldsymbol{p}} \in [0,k]$.

Combining Corollary 2 and Theorem 5 together with Lemma 4 and Lemma 5 yields the following convergence result:

Theorem 6 *Suppose Δt is sufficiently small, and that the solution $(v, \boldsymbol{p})$ of the mixed hyperbolic problem (2) is sufficiently smooth in space and time that the estimates of Lemma 4 and Lemma 5 can be applied. In addition suppose $v_{h,\Delta t}$, $\boldsymbol{p}_{h,\Delta t}$ and $\widehat{v}_{h,\Delta t}$ satisfy the CTG-HDG problem (9) using the non-dissipative numerical flux (11) with the initial conditions*

$$v_{h,\Delta t}(\cdot, 0) = \Pi_W v_0 \quad and \quad \boldsymbol{p}_{h,\Delta t}(\cdot, 0) = \boldsymbol{\Pi_V} \boldsymbol{p}_0 \qquad in \; \Omega$$

and $\widehat{v}_{h,\Delta t}(\cdot, 0) = P_M v_0|_{\partial \mathcal{T}_h}$ on $\partial \mathcal{T}_h$. Then the following error estimate holds:

$$\max_{0 \leq n \leq N} \left(\|v(\cdot, t_n) - v_{h,\Delta t}(\cdot, t_n)\|_\Omega + \|\boldsymbol{p}(\cdot, t_n) - \boldsymbol{p}_{h,\Delta t}(\cdot, t_n)\|_\Omega \right)$$
$$\leq CT^{1/2} \left(h^{\ell_v+1/2} M_1(T) + h^{\ell_{\boldsymbol{p}}+1/2} M_2(T) + \Delta t^{q+1} M_3(T) \right),$$
$$\|v - v_{h,\Delta t}\|_{L^2([0,T]; L^2(\Omega))} + \|\boldsymbol{p} - \boldsymbol{p}_{h,\Delta t}\|_{L^2([0,T]; \boldsymbol{L}^2(\Omega))}$$
$$\leq CT \left(h^{\ell_v+1/2} M_1(T) + h^{\ell_{\boldsymbol{p}}+1/2} M_2(T) + \Delta t^{q+1} M_3(T) \right),$$

for $\ell_v, \ell_{\boldsymbol{p}} \in [0, k]$, where

$$M_1(T) \leq \|v\|_{H^1([0,T];H^{\ell_v+1}(\Omega))},$$

$$M_2(T) \leq \|\nabla \cdot \boldsymbol{p}\|_{H^1([0,T];H^{\ell_{\boldsymbol{p}}+1}(\Omega))} + h^{1/2}\|\dot{\boldsymbol{p}}\|_{L^2([0,T];\boldsymbol{H}^{\ell_{\boldsymbol{p}}+1}(\Omega))},$$

$$M_3(T) \leq \|\nabla v\|_{H^{q+1}([0,T];\boldsymbol{L}^2(\Omega))} + \|\nabla \cdot \boldsymbol{p}\|_{H^{q+1}([0,T];L^2(\Omega))}.$$

6 Numerical results

In this section we probe the error estimates for the dissipative scheme and for the non-dissipative scheme established in Theorem 3 and Theorem 6 by numerical examples. We consider the lowest order case $q = 1$ in time when the method (9) corresponds to the implicit midpoint rule. On an equidistant grid $t_n = n\Delta t$, $0 \leq n \leq N$, on the time interval $[0, T]$ with step size Δt that is determined depending on the mesh parameter h of the spatial mesh the numerical approximations $v_{h,\Delta t}^n := v_{h,\Delta t}(\cdot, t_n) \in W_h$, $\boldsymbol{p}_{h,\Delta t}^n := \boldsymbol{p}_{h,\Delta t}(\cdot, t_n) \in \boldsymbol{V}_h$ and $\widehat{v}_{h,\Delta t}^n := \widehat{v}_{h,\Delta t}(\cdot, t_n) \in M_h$ satisfy the system of equations

$$(\delta_{\Delta t} v_{h,\Delta t}^n, \xi)_{\mathcal{T}_h} = -(\overline{\boldsymbol{p}}_{h,\Delta t}^n, \nabla \xi)_{\mathcal{T}_h} + \langle \overline{\widehat{\boldsymbol{p}}}_{h,\Delta t}^n \cdot \boldsymbol{n}, \xi \rangle_{\partial \mathcal{T}_h} + (f^{n+1/2}, \xi)_{\mathcal{T}_h}, \tag{33a}$$

$$(\delta_{\Delta t} \boldsymbol{p}_{h,\Delta t}^n, \boldsymbol{\psi})_{\mathcal{T}_h} = -(\overline{v}_{h,\Delta t}^n, \nabla \cdot \boldsymbol{\psi})_{\mathcal{T}_h} + \langle \overline{\widehat{v}}_{h,\Delta t}^n, \boldsymbol{\psi} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h}, \tag{33b}$$

$$\langle \overline{\widehat{\boldsymbol{p}}}_{h,\Delta t}^n \cdot \boldsymbol{n}, \mu \rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} = 0, \tag{33c}$$

$$\langle \overline{\widehat{v}}_{h,\Delta t}^n, \eta \rangle_{\partial \Omega} = \langle \dot{g}^{n+1/2}, \eta \rangle_{\partial \Omega}, \tag{33d}$$

for all $\xi \in W_h$, $\boldsymbol{\psi} \in \boldsymbol{V}_h$ and $\mu, \eta \in M_h$, where

$$f^{n+1/2} := f(\cdot, t_n + \Delta t/2) \qquad \text{and} \qquad \dot{g}^{n+1/2} := \dot{g}(\cdot, t_n + \Delta t/2),$$

as well as

$$\delta_{\Delta t} v_{h,\Delta t}^n := \frac{v_{h,\Delta t}^n - v_{h,\Delta t}^{n-1}}{\Delta t} \qquad \text{and} \qquad \overline{v}_{h,\Delta t}^n := \frac{v_{h,\Delta t}^n + v_{h,\Delta t}^{n-1}}{2},$$

and similarly for $\boldsymbol{p}_{h,\Delta t}$. Accordingly the numerical flux (10) corresponding to the dissipative scheme reduces to

$$\overline{\widehat{\boldsymbol{p}}}_{h,\Delta t}^n = \overline{\boldsymbol{p}}_{h,\Delta t}^n - \tau(\overline{v}_{h,\Delta t}^n - \overline{\widehat{v}}_{h,\Delta t}^n)\boldsymbol{n} \qquad \text{on } \partial \mathcal{T}_h, \tag{34}$$

whereas the numerical flux (11) becomes

$$\overline{\widehat{\boldsymbol{p}}}_{h,\Delta t}^n = \overline{\boldsymbol{p}}_{h,\Delta t}^n - \tau(\delta_{\Delta t} v_{h,\Delta t}^n - \delta_{\Delta t} \widehat{v}_{h,\Delta t}^n)\boldsymbol{n} \qquad \text{on } \partial \mathcal{T}_h. \tag{35}$$

We have implemented (33) together with (34) and (35), respectively, for piecewise constant, piecewise linear and piecewise quadratic Lagrange elements in $\mathbb{R}^2$. The local mass and stiffness matrices as well as the right hand side are evaluated using order 6 quadrature rules on the triangles as well as on the edges.

Example 1 We consider a square domain $\Omega = (-1, 1) \times (-1, 1) \subset \mathbb{R}^2$, homogeneous boundary conditions $g = 0$ on $\partial \Omega \times (0, T)$, a homogeneous source term $f = 0$ in $\Omega \times (0, T)$ and initial conditions $v_0 = \sin(\pi x_1)\sin(\pi x_2)$ as well as $\boldsymbol{p}_0 = [0, 0]^T$ in Ω, where $\boldsymbol{x} = [x_1, x_2]^T$
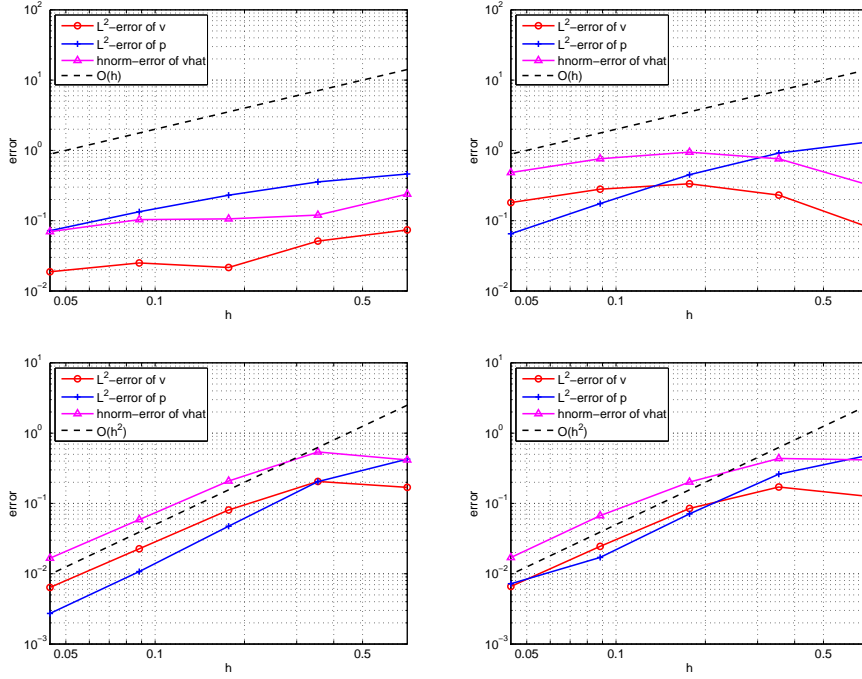
Fig. 1 Errors $\|(v - v_{h,\Delta t})(\cdot, 1)\|_\Omega$, $\|(\boldsymbol{p} - \boldsymbol{p}_{h,\Delta t})(\cdot, 1)\|_\Omega$ and $\|(v - \widehat{v}_{h,\Delta t})(\cdot, 1)\|_h$ with $k = 0$ and $k = 1$ (top down) for the dissipative flux (34) (left) and non-dissipative flux (35) (right).

and as usual $\boldsymbol{x}^T$ denotes the transpose of $\boldsymbol{x}$. The corresponding exact solution of (2) is given by

$$v(\boldsymbol{x},t) = \cos(\sqrt{2}\pi t)\sin(\pi x_1)\sin(\pi x_2),$$

$$\boldsymbol{p}(\boldsymbol{x},t) = \frac{1}{\sqrt{2}}\sin(\sqrt{2}\pi t)\left[\cos(\pi x_1)\sin(\pi x_2), \sin(\pi x_1)\cos(\pi x_2)\right]^T.$$

This example has also been used as a test problem in [16]. In contrast to the present work Nguyen et al. discuss the dissipative scheme only and use higher-order backward difference formulas as well as diagonally implicit Runge-Kutta methods for time discretization.

Figure 1 shows the absolute error at time $T = 1$ in the scalar variable $\|(v - v_{h,\Delta t})(\cdot, 1)\|_\Omega$, in the flux variable $\|(\boldsymbol{p} - \boldsymbol{p}_{h,\Delta t})(\cdot, 1)\|_\Omega$ and in the numerical trace $\|(v - \widehat{v}_{h,\Delta t})(\cdot, 1)\|_h$ as a function of the mesh parameter h for polynomial degree $k = 0$ and $k = 1$ (top down). The plots in the left column correspond to the dissipative scheme, i.e., to the numerical trace (34), while the right column has been obtained with the non-dissipative scheme, i.e., using the numerical trace (35). Here $\|\cdot\|_h$ denotes a mesh dependent norm on $L^2(\partial\mathcal{T}_h)$ given by $\|\mu\|_h^2 := \sum_{K \in \mathcal{T}_h} h_K \|\mu\|_{\partial K}^2$ for any $\mu \in L^2(\partial\mathcal{T}_h)$. For the space discretization we use structured meshes $\mathcal{T}_h$ with $M = 8, 32, 128, 512$ and 2048 triangles, as shown in Figure 2 (left) for $M = 32$, and accordingly we choose the time step size $\Delta t = h$. In all our examples the coupling parameter is set to be $\tau = 1$.

Since the time discretization used in this example is second order accurate only, we use smaller time steps $\Delta t = h^{3/2}$ instead of $\Delta t = h$ in the corresponding plots for polynomial degree $k = 2$ in space shown in Figure 3.
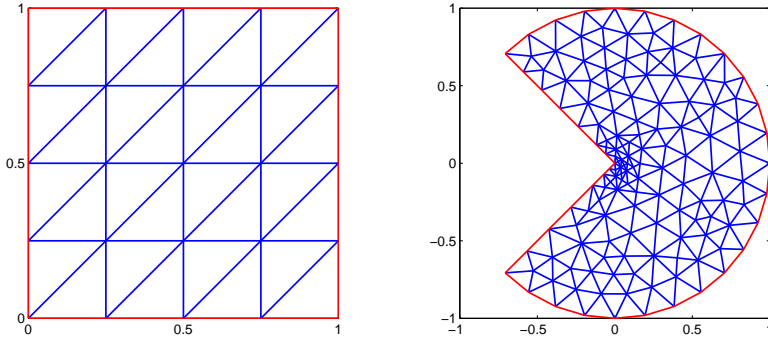
**Fig. 2** Structured mesh of Example 1 (left) and the locally refined unstructured mesh of Example 2 (right).
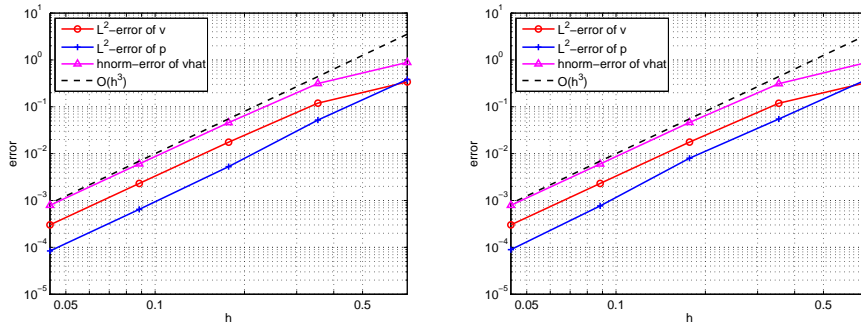
Fig. 3 Errors $\|(v - v_{h,\Delta t})(\cdot, 1)\|_{\Omega}$, $\|(\boldsymbol{p} - \boldsymbol{p}_{h,\Delta t})(\cdot, 1)\|_{\Omega}$ and $\|(v - \widehat{v}_{h,\Delta t})(\cdot, 1)\|_h$ with $k = 2$ for the dissipative flux (34) (left) and the non-dissipative flux (35) (right).

After a pre-asymptotic region of slower convergence, which is more pronounced for lower order elements in space as well as for the non-dissipative flux, we observe optimal convergence for both numerical fluxes. This behavior is actually better than our theoretical results predict: While Theorem 3 guarantees optimal convergence for the dissipative scheme, we did not prove optimal order convergence for the non-dissipative scheme, missing by a factor $h^{1/2}$. However, as can be seen in Figure 3 the convergence behavior of the two schemes is almost identical for $k = 2$.   ◇

Example 2 In the second example we consider unstructured locally refined grids with $M = 197$, 254, 417, 629 and 1034 triangles on a non-convex domain as shown in Figure 2 (right) for $M = 254$. This example is intended to show that the CDG-HDG method can be used on locally refined grids with a time step size that is chosen consistent with the largest element of the grid. The boundary conditions and the source term in (9) are chosen such that the exact solution of (2) is given by the time-harmonic wave

$$v(\boldsymbol{x}, t) = \cos(\omega t) J_{\xi}\left(\kappa \sqrt{x_1^2 + x_2^2}\right) \cos\left(\xi \arctan(x_2/x_1)\right)$$

with $\omega = \kappa = 4$ and $\xi = 2/3$, where J_{ξ} denotes the Bessel function of the first kind of order ξ. A similar example has been used as a test problem for a corresponding HDG method for
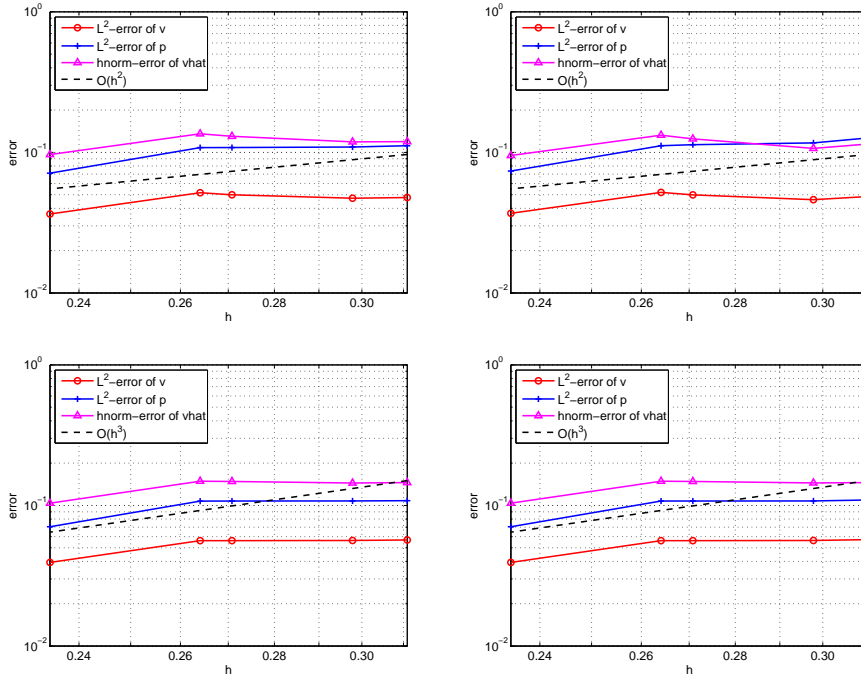
Fig. 4 Errors $\|(v - v_{h,\Delta t})(\cdot,1)\|_{\Omega}$, $\|(\boldsymbol{p} - \boldsymbol{p}_{h,\Delta t})(\cdot,1)\|_{\Omega}$ and $\|(v - \widehat{v}_{h,\Delta t})(\cdot,1)\|_h$ with $k = 1$ and $k = 2$ (top down) for the dissipative flux (34) (left) and the non-dissipative flux (35) (right).

the Helmholtz equation in [7]. If $\cos(\omega t) \neq 0$, the exact solution $v(\cdot,t)$ has a singularity at $(0,0)$ such that $v(\cdot,t) \in H^1(\Omega)$ but $v(\cdot,t) \notin H^2(\Omega)$.

Figure 4 shows the absolute errors at final time $\|(v - v_{h,\Delta t})(\cdot,1)\|_{\Omega}$, $\|(\boldsymbol{p} - \boldsymbol{p}_{h,\Delta t})(\cdot,1)\|_{\Omega}$ and $\|(v - \widehat{v}_{h,\Delta t})(\cdot,1)\|_h$ as a function of the mesh parameter h for polynomial degree $k = 1$ and $k = 2$ (top down). As before, the left column corresponds to the dissipative scheme while the right column has been obtained with the non-dissipative scheme. The time step size in this example is chosen to be $\Delta t = h$, i.e., according to the diameter of the largest triangle in the spatial mesh. Due to the strongly local refinement of the grids near the origin the mesh parameter h varies much less in this example than in the previous one.

The results in Figure 4 show that the method converges at the optimal rate and so indicates that the method is able to handle locally refined grids as intended.   ◇

Further numerical results for the dissipative scheme (6)–(7) using a different time stepping can be found in [16]. For more details on the numerical implementation of HDG-schemes, in particular on hybridization strategies and post-processing of the numerical solution we refer to [3, 15, 16].

7 Concluding Remarks

We have provided an error analysis of two fully discrete methods for approximating the acoustic wave equation using an arbitrary order implicit scheme in time. The resulting

CTG-HDG schemes are unconditionally stable. Although our theoretical analysis provides a sub-optimal convergence rate for our new non-dissipative scheme, numerical results do not indicate any loss of order, so an interesting problem is to remedy or explain this discrepancy.

The main disadvantages of the CTG-HDG scheme is that as the order of convergence in time increases, larger and larger discrete problems need to be solved. Other Runge-Kutta schemes such as the DIRK scheme used in [15] can avoid this, although it is still necessary to perform an implicit solve for each stage. Extensions of the analysis to include this case would also be useful.

A Projection estimate on the element boundaries

In this section we give a proof of the projection estimate on the element boundaries (4c). Our arguments rest on those used in [4] to prove the corresponding interior estimates (4a) and (4b).

We start with a slight modification of Lemma A.2 of [4].

Lemma 6 *Let $K \in \mathcal{T}_h$, τ as introduced in Section 2.1, and suppose*

$$p \in \mathcal{P}_k^\perp(K) := \{\phi \in \mathcal{P}_k(K) \mid (\phi, w)_K = 0 \text{ for all } w \in \mathcal{P}_{k-1}(K)\}$$

satisfies

$$\langle \tau p, \phi \rangle_{\partial K} = b(\phi) \qquad \text{for all } \phi \in P_k^\perp(K),$$

where $b : \mathcal{P}_k^\perp(K) \to \mathbb{R}$ is linear. Then

$$\|\tau p\|_{\partial K} \le C h_K^{1/2} \|b\|,$$

where $\|b\|$ denotes the operator norm of b with respect to the L^2-norm on $\mathcal{P}_k^\perp(K)$.

Proof Denoting by F the edge/face of K at which $\tau = \tau_K^{\max}$,

$$\|\tau p\|_{\partial K}^2 = \langle \tau p, \tau p \rangle_{\partial K} \le \tau_K^{\max} \langle \tau p, p \rangle_{\partial K} = \tau_K^{\max} b(p) \le \tau_K^{\max} \|b\| \|p\|_K.$$

Using the estimate

$$\|p\|_K \le C h_K^{1/2} \|p\|_F \qquad \text{for all } p \in \mathcal{P}_K^\perp(K),$$

which has been shown in Lemma A.1 of [4], gives

$$\|\tau p\|_{\partial K}^2 \le \tau_K^{\max} \|b\| C h_K^{1/2} \|p\|_F = \|b\| C h_K^{1/2} \|\tau_K^{\max} p\|_F \le C h_K^{1/2} \|b\| \|\tau p\|_{\partial K}.$$

$\square$

The following proposition should be compared to Proposition A.2 of [4].

Proposition 1 *Suppose $k \ge 0$, and let $K \in \mathcal{T}_h$ and τ as introduced in Section 2.1. Then,*

$$\|\tau(\Pi_W w - w)\|_{\partial K} \le C h_K^{1/2} \left(h_K^{\ell_z} |\nabla \cdot z|_{H^{\ell_z}(K)} + \tau_K^{\max} h_K^{\ell_w} |w|_{H^{\ell_w+1}(K)} \right)$$

for $\ell_w, \ell_z \in [0, k]$.

Proof Denoting $\delta^w := \Pi_W w - w_k$, where w_k is the L^2-projection of w onto $\mathcal{P}_k(K)$, we have

$$\|\tau(\Pi_W w - w)\|_{\partial K} \le \|\tau(w - w_k)\|_{\partial K} + \|\tau \delta^w\|_{\partial K}.$$

Applying a trace inequality and the approximation properties of the L^2-projection the first term can be estimated by

$$\|\tau(w - w_k)\|_{\partial K} \le \tau_K^{\max} \|w - w_k\|_{\partial K} \le \tau_K^{\max} h_K^{-1/2} \left(\|w - w_k\|_K + h_K |w - w_k|_{H^1(K)} \right)$$

$$\le C \tau_K^{\max} h_K^{\ell_w + 1/2} |w|_{H^{\ell_w+1}(K)}.$$

To estimate the second term we recall that on each element $K \in \mathcal{T}_h$ the component $\Pi_W w$ satisfies

$$(\Pi_W w, \phi)_K = (w, \phi)_K \qquad \text{for all } \phi \in P_{k-1}(K),$$
$$\langle \tau \Pi_W w, \phi \rangle_{\partial K} = (\nabla \cdot z, \phi)_K + \langle \tau w, \phi \rangle_{\partial K} \qquad \text{for all } \phi \in P_k^\perp(K)$$

(see Proposition A.1 of [4]). This implies that $\delta^w \in \mathcal{P}_k^\perp$ and

$$\langle \tau \delta^w, \phi \rangle_{\partial K} = b_w(\phi) + b_z(\phi) \qquad \text{for all } \phi \in \mathcal{P}_k^\perp(K),$$

where $b_w(\phi) := \langle \tau(w - w_k), \phi \rangle_{\partial K}$ and $b_z(\phi) := (\nabla \cdot z, \phi)_K$. It has been shown in the proof of Proposition A.2 in [4] that

$$\|b_w\| \leq C\tau_K^{\max} h_K^{\ell_w} |w|_{H^{\ell_w+1}(K)} \quad \text{and} \quad \|b_z\| \leq Ch_K^{\ell_z} |\nabla \cdot z|_{H^{\ell_z}(K)}.$$

Since by Lemma 6,

$$\|\tau \delta^w\|_{\partial K} \leq Ch_K^{1/2}(\|b_w\| + \|b_z\|),$$

this ends the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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