Localizing acoustic and electromagnetic waves in space and time

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Abstract

We study time-dependent acoustic and electromagnetic waves governed by the scalar wave equation or Maxwell's equations in a bounded three-dimensional domain. We establish the existence of time-dependent boundary excitations that can be prescribed on any open subset of the boundary of the domain such that the associated waves are strongly localized in space in the sense that they possess arbitrarily large norms in a given subdomain and on a given time-interval, while remaining arbitrarily small in any other given subdomain for all times. Similarly, we also show the existence of boundary data such that the associated waves are strongly localized in time in the sense that they possess arbitrarily large norms in a given subdomain and on a given time-interval, while remaining arbitrarily small on the same subdomain but on any other prescribed time-interval. In case that we have access to the possibly inhomogeneous coefficients in the wave equation or in the Maxwell system, we also give explicit constructions to obtain boundary data that generate these localized waves, and we comment on possible applications.

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1 Introduction.

Localized or singular solutions have proven to be a fundamental tool in the study of uniqueness results and reconstruction algorithms for inverse boundary value problems for partial differential equations. Already very early uniqueness proofs for the inverse conductivity problem in [23, 24] rely on probing the unknown conductivity distribution using sequences of boundary potentials that generate highly focused voltage distributions inside the domain. One particular construction of such localized potentials for elliptic equations, which has been introduced in [8], combines a duality argument for certain data-to-solution operators with a unique continuation principle for solutions of the underlying partial differential equation to establish the existence of sequences of boundary currents that give rise to potentials inside the domain possessing arbitrarily large norm on a prescribed subdomain, while nearly vanishing on another given subdomain. These localized potentials have been successfully applied to establish novel uniqueness results for inverse boundary value problems from local boundary data in [11, 16]. The combination of such localized potentials with monotonicity relations for the associated Neumann-to-Dirichlet operators has also been used to establish the theoretical foundation of novel qualitative reconstruction methods for the inverse conductivity problem in [15, 30, 31].

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Meanwhile, these localized potentials and the monotonicity based uniqueness proofs and reconstruction methods for the inverse conductivity problem have been extended to time-harmonic wave equations describing acoustic, electromagnetic and elastic waves at fixed frequency in [5, 12, 13, 14]. It has also been noted that the construction of localized solutions in the fixed frequency case is connected to the study of quantitative Runge estimates as considered in [27, 29]. Corresponding localized waves and monotonicity based reconstruction methods for time-harmonic inverse scattering problems on unbounded domains have been analyzed in [1, 9, 17].

The aim of this work is to extend the concept of localized potentials from [11, 12, 14] to initial boundary value problems for the time-dependent wave equation and for time-dependent Maxwell's equations. Since time offers an additional degree of freedom, we consider the following two cases:

(i) *Focusing in space:* We show that there exist time-dependent excitations on some part of the boundary of the domain that generate waves in the interior of the domain possessing arbitrarily large norms in a prescribed subdomain and on a given time-interval, while nearly vanishing in another prescribed subdomain for all times.

(ii) *Focusing in time:* We establish the existence of time-dependent boundary excitations on some part of the boundary of the domain that generate waves in the interior of the domain possessing arbitrarily large norms in a prescribed subdomain and on a given time-interval, while nearly vanishing on the same subdomain but on another prescribed time-interval.

Our proofs combine duality arguments for data-to-solution operators with unique continuation principles for solutions to the time-dependent wave equation and Maxwell's equations. In particular, we invoke Tataru's seminal result [32, 33] on global unique continuation for the wave equation in optimal time, which generalized earlier important contributions from [18] and [28]. For an introduction to such unique continuation results and related applications, we ask the reader to consult [19] and [26]. For some principally diagonalizable systems, the unique continuation result in [32] was extended in [6, 7]. This is what we use in our results for Maxwell's equations. However, it requires stronger smoothness assumptions on the electric permittivity and the magnetic permeability than those needed for the rest of the argument. Moreover, the finite speed of propagation of solutions to the wave equation and to Maxwell's equations leads to natural restrictions on the time-intervals for rising and perhaps disappearing of the localized waves in terms of the distances of the respective subdomains from the part of the boundary, where the boundary values are excited. As usual, these distances are to be measured with respect to the travel time metric.

In case that we have access to the possibly inhomogeneous parameters in the wave equation or in the time-dependent Maxwell system, we also provide explicit constructions of the boundary excitations that can be used to generate localized waves. These constructions can immediately be translated into numerical algorithms. In this context, we also note that iterated time reversal has been applied in [3, 4, 20] to develop algorithms that use observations of the hyperbolic Dirichlet-to-Neumann map for the wave equation to focus scalar waves to a delta distribution at some fixed times even in unknown media. However, then the focusing is done in travel time coordinates. An advantage of our results might be the possibility to prescribe regions away from the focus area where the amplitude of the waves are kept at arbitrary small levels for all times.

Besides possible applications of localized solutions for time-dependent wave or Maxwell's equations in uniqueness proofs for inverse boundary value problems or in monotonicity-based qualitative reconstruction methods, which have been our main motivation for this work, several other potential applications have for instance been proposed in [4, 12, 20]. These include ultrasound induced heating, inductive charging, or secure communication.

The remainder of this article proceeds as follows. In Section 2 we introduce our notation and the two initial boundary value problems (IBVP) for the wave equation and for Maxwell's equations that we use as basic models for acoustic and electromagnetic wave propagation throughout this work. We also briefly discuss the well-posedness of these problems. Then, in Section 3, we establish the existence of localized solutions for the IBVP for the wave equation and comment on their construction. Localized solutions for the IBVP for Maxwell's equations are developed in Section 4. In the Appendix we collect some abstract functional analytic results that are used in Sections 3 and 4.

2 The mathematical setting.

We start by introducing some notation concerning the two IBVP that we study in the rest of this work. Let $T > 0$ and let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^2-smooth boundary $\partial\Omega$. Furthermore, let $\Gamma \subseteq \partial\Omega$ be a relatively open subset that will represent the support of all boundary excitations in the following.

The first problem that we discuss is an acoustic wave equation. Denoting by $c \in C^1(\overline{\Omega})$ and $q \in L^\infty(\Omega)$ the wave speed and an external potential, respectively, we consider the equation

$$c^{-2}(x)\partial_t^2 u_f - \Delta u_f + q(x)u_f \;=\; 0 \qquad \text{in } \Omega \times (0,T)\,, \tag{2.1a}$$

together with the homogeneous initial conditions

$$u_f|_{t=0} \;=\; \partial_t u_f|_{t=0} \;=\; 0 \qquad \text{in } \Omega\,, \tag{2.1b}$$

and the Dirichlet boundary condition

$$u_f|_{\partial\Omega\times(0,T)} \;=\; f \text{ on } \Gamma \times (0,T) \quad \text{and} \quad u_f|_{\partial\Omega\times(0,T)} \;=\; 0 \text{ on } (\partial\Omega \setminus \Gamma) \times (0,T) \tag{2.1c}$$

for some $f \in L^2(\Gamma \times (0,T))$.

Throughout we assume that the wave speed is bounded away from zero; that is $c \geq c_-$ in Ω for some positive constant $c_- > 0$. Following [19, p. 72], we call $u_f \in L^2(\Omega \times (0,T))$ a weak solution of the IBVP (2.1) if, for any $\psi \in H^2(\Omega \times (0,T))$ with $\psi|_{\Gamma\times(0,T)} = 0$ and $\psi|_{t=T} = \partial_t\psi|_{t=T} = 0$,

$$\int_0^T \int_\Omega u_f \big(c^{-2}(x)\partial_t^2\psi - \Delta\psi + q(x)\psi\big) \,\mathrm{d}x \,\mathrm{d}t \;=\; \int_0^T \int_\Gamma f \partial_{\boldsymbol{\nu}}\psi \,\mathrm{d}s_x \,\mathrm{d}t\,.$$

As usual $\boldsymbol{\nu}$ denotes the unit outward normal on $\partial\Omega$ and $\partial_{\boldsymbol{\nu}}$ the associated normal derivative.

Proposition 2.1. *For any $f \in L^2(\Gamma \times (0,T))$, there exists a unique weak solution u_f of (2.1). This solution satisfies $u_f \in C([0,T]; L^2(\Omega))$, and it depends continuously on the data.*

Proof. Under the aforementioned hypotheses this result can be inferred from Theorem 2.30 and Lemma 2.42 in [19] (see also [25]). □

The second problem that we consider is a time-dependent Maxwell system. Throughout, we denote $\boldsymbol{L}^2(\Omega) := L^2(\Omega;\mathbb{R}^3)$ and

$$\boldsymbol{H}(\mathbf{curl};\Omega) \;:=\; \left\{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega) \mid \mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)\right\}\,.$$

with the norms defined accordingly. We also recall the tangential trace operators

$$\begin{aligned} \gamma_\tau &: \boldsymbol{H}(\mathbf{curl};\Omega) \to H^{-1/2}(\mathrm{Div};\partial\Omega)\,, \quad \boldsymbol{u} \mapsto \boldsymbol{\nu} \times \boldsymbol{u}|_{\partial\Omega}\,,\\ \pi_\tau &: \boldsymbol{H}(\mathbf{curl};\Omega) \to H^{-1/2}(\mathrm{Curl};\partial\Omega)\,, \quad \boldsymbol{u} \mapsto \boldsymbol{\nu} \times (\boldsymbol{u} \times \boldsymbol{\nu})|_{\partial\Omega}\,. \end{aligned}$$

For a detailed discussion on this, and in particular on the trace spaces $H^{-1/2}(\mathrm{Div};\partial\Omega)$ and $H^{-1/2}(\mathrm{Curl};\partial\Omega)$, we refer to [21, Sec. 5.1].

Denoting by $\varepsilon,\mu \in L^\infty(\Omega)$ the electric permittivity and the magnetic permeability, respectively, we consider the system of equations

$$\varepsilon(x)\,\partial_t \boldsymbol{E_f} - \mathbf{curl}\,\boldsymbol{H_f} \;=\; 0\,, \quad \mu(x)\,\partial_t \boldsymbol{H_f} + \mathbf{curl}\,\boldsymbol{E_f} \;=\; 0 \qquad \text{in } \Omega\times(0,T)\,, \tag{2.2a}$$

together with the homogeneous initial conditions

$$\boldsymbol{E_f}|_{t=0} \;=\; \boldsymbol{H_f}|_{t=0} \;=\; 0 \qquad \text{in } \Omega\,, \tag{2.2b}$$

and the boundary condition

$$\boldsymbol{\nu}\times\boldsymbol{E_f}|_{\partial\Omega\times(0,T)} \;=\; \boldsymbol{f} \text{ on } \Gamma\times(0,T) \quad \text{and} \quad \boldsymbol{\nu}\times\boldsymbol{E_f}|_{\partial\Omega\times(0,T)} \;=\; 0 \text{ on } (\partial\Omega\setminus\Gamma)\times(0,T) \tag{2.2c}$$

for some $\boldsymbol{f} \in \mathcal{H}_0^1([0,T];\widetilde{H}^{-1/2}(\mathrm{Div};\Gamma))$. Here $\widetilde{H}^{-1/2}(\mathrm{Div};\Gamma)$ we denotes the closure of $C_0^2(\Gamma)$ with respect to $\|\cdot\|_{H^{-1/2}(\mathrm{Div};\partial\Omega)}$. Furthermore, for any Hilbert space X we use the notation

$$\mathcal{H}_0^1([0,T];X) \;:=\; \big\{\boldsymbol{f}\in H^1((0,T);X) \;\big|\; \boldsymbol{f}(0)=0\big\}\,.$$

In view of the vanishing initial condition, we consider the following inner product on $\mathcal{H}_0^1([0,T];X)$,

$$\langle \boldsymbol{f},\boldsymbol{g}\rangle_{\mathcal{H}_0^1([0,T];X)} \;:=\; \int_0^T \big\langle \partial_t\boldsymbol{f}(t),\partial_t\boldsymbol{g}(t)\big\rangle_X \,\mathrm{d}t \qquad \text{for all } \boldsymbol{f},\boldsymbol{g}\in\mathcal{H}_0^1([0,T];X)\,,$$

where $\langle\cdot,\cdot\rangle_X$ denotes the inner product of X. For the rest of this article, we suppress explicit reference to the underlying function space in the notation $\langle\cdot,\cdot\rangle$; it will be understood from the context.

Throughout we assume that the electric permittivity and the magnetic permeability are bounded away from zero; that is $\varepsilon \ge \varepsilon_-$ and $\mu \ge \mu_-$ a.e. on Ω for some positive constants $\varepsilon_-,\mu_- > 0$. Following [2], we call $(\boldsymbol{E_f},\boldsymbol{H_f})$ with[1]

$$\begin{aligned}\boldsymbol{E_f} &\in \mathcal{H}_0^1([0,T];\boldsymbol{H}_0(\mathbf{curl};\Omega)^*)\cap L^\infty((0,T);\boldsymbol{L}^2(\Omega))\,,\\ \boldsymbol{H_f} &\in \mathcal{H}_0^1([0,T];\boldsymbol{H}(\mathbf{curl};\Omega)^*)\cap L^\infty([0,T];\boldsymbol{L}^2(\Omega))\,,\end{aligned}$$

a weak solution to the IBVP (2.2) if, for all $t\in(0,T)$, it satisfies

$$\big\langle\varepsilon(x)\partial_t\boldsymbol{E_f}(t),\boldsymbol{\Phi}\big\rangle - \int_\Omega \boldsymbol{H_f}(t)\cdot\mathbf{curl}\,\boldsymbol{\Phi}\,\mathrm{d}x \;=\; 0\,, \qquad \text{for all } \boldsymbol{\Phi}\in\boldsymbol{H}_0(\mathbf{curl};\Omega)\,, \tag{2.3a}$$

$$\big\langle\mu(x)\partial_t\boldsymbol{H_f}(t),\boldsymbol{\Psi}\big\rangle + \int_\Omega \boldsymbol{E_f}(t)\cdot\mathbf{curl}\,\boldsymbol{\Psi}\,\mathrm{d}x \;=\; -\,\langle\boldsymbol{f}(t),\pi_\tau[\boldsymbol{\Psi}]\rangle \quad \text{for all } \boldsymbol{\Psi}\in\boldsymbol{H}(\mathbf{curl};\Omega)\,. \tag{2.3b}$$

Let us underline that $\langle\cdot,\cdot\rangle$ on the right hand side of the second equation in (2.3) denotes the dual pairing between $H^{-1/2}(\mathrm{Div};\partial\Omega)$ and $H^{-1/2}(\mathrm{Curl};\partial\Omega)$. However, the same notation appearing on the left hand side of the first and second equation of (2.3) denotes the dual pairing between $\boldsymbol{H}_0(\mathbf{curl};\Omega)^*$ and $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ as well as $\boldsymbol{H}(\mathbf{curl};\Omega)^*$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$, respectively.

Proposition 2.2. *For any $\boldsymbol{f}\in\mathcal{H}_0^1([0,T];\widetilde{H}^{-1/2}(\mathrm{Div};\Gamma))$, there exists a unique weak solution $(\boldsymbol{E_f},\boldsymbol{H_f})$ of (2.2). This solution satisfies $(\boldsymbol{E_f},\boldsymbol{H_f})\in C([0,T];\boldsymbol{L}^2(\Omega)^2)$, and it depends continuously on the data. Furthermore, we have $\mathrm{div}(\varepsilon(x)\boldsymbol{E_f}(t)) = \mathrm{div}(\mu(x)\boldsymbol{H_f}(t)) = 0$ for all $t\in[0,T]$.*

[1]For a given Banach space X, we denote its topological dual by X^*, and accordingly $\langle\cdot,\cdot\rangle_X$ also denotes the duality pairing between X^* and X.

Proof. Thanks to the surjectivity of the tangential trace $\gamma_\tau : H(\mathbf{curl};\Omega) \to H^{-1/2}(\mathrm{Div};\partial\Omega)$, we can consider a lifting operator $G_\tau : \mathcal{H}_0^1([0,T];\widetilde{H}^{-1/2}(\mathrm{Div};\Gamma)) \mapsto \mathcal{H}_0^1([0,T];\boldsymbol{H}(\mathbf{curl};\Omega))$ satisfying

$$\gamma_\tau[G_\tau[\boldsymbol{f}](t)] \;=\; \boldsymbol{f}(t) \qquad \text{for all } t \in (0,T)\,.$$

In view of this, we define $\widetilde{\boldsymbol{E}}_{\boldsymbol{f}} = \boldsymbol{E}_{\boldsymbol{f}} - G_\tau[\boldsymbol{f}]$ and recast (2.3), for all $t \in (0,T)$, to the following weak formulation

$$\big\langle \varepsilon(x)\partial_t \widetilde{\boldsymbol{E}}_{\boldsymbol{f}}(t), \boldsymbol{\Phi}\big\rangle - \int_\Omega \boldsymbol{H}_{\boldsymbol{f}}(t)\cdot \mathbf{curl}\,\boldsymbol{\Phi}\,\mathrm{d}x \;=\; -\int_\Omega \varepsilon(x)\partial_t G_\tau[\boldsymbol{f}]\cdot\boldsymbol{\Phi}\,\mathrm{d}x\,, \tag{2.4a}$$

$$\big\langle \mu(x)\partial_t \boldsymbol{H}_{\boldsymbol{f}}(t), \boldsymbol{\Psi}\big\rangle + \int_\Omega \widetilde{\boldsymbol{E}}_{\boldsymbol{f}}(t)\cdot \mathbf{curl}\,\boldsymbol{\Psi}\,\mathrm{d}x \;=\; -\int_\Omega \mathbf{curl}\, G_\tau[\boldsymbol{f}]\cdot\boldsymbol{\Psi}\,\mathrm{d}x\,, \tag{2.4b}$$

for all $\boldsymbol{\Phi} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ and $\boldsymbol{\Psi} \in \boldsymbol{H}(\mathbf{curl};\Omega)$. Still imposing homogeneous initial conditions $\widetilde{\boldsymbol{E}}_{\boldsymbol{f}}|_{t=0} = \boldsymbol{H}_{\boldsymbol{f}}|_{t=0} = 0$, the existence and uniqueness of a solution $(\widetilde{\boldsymbol{E}}_{\boldsymbol{f}}, \boldsymbol{H}_{\boldsymbol{f}})$ with

$$\widetilde{\boldsymbol{E}}_{\boldsymbol{f}} \in \mathcal{H}_0^1([0,T];\boldsymbol{H}_0(\mathbf{curl};\Omega)^*) \cap L^\infty((0,T);\boldsymbol{L}^2(\Omega))\,,$$
$$\boldsymbol{H}_{\boldsymbol{f}} \in \mathcal{H}_0^1([0,T];\boldsymbol{H}(\mathbf{curl};\Omega)^*) \cap L^\infty((0,T);\boldsymbol{L}^2(\Omega))\,,$$

to (2.4) can be established using the Galerkin method, as done in [2, Thm. 2], or using semi-group theory (see [22, Thm. 5.3] for such a result under slightly stronger regularity assumptions for the source term). This in turn implies that the IBVP (2.2) admits a unique weak solution $(\boldsymbol{E}_{\boldsymbol{f}}, \boldsymbol{H}_{\boldsymbol{f}}) \in C([0,T];\boldsymbol{L}^2(\Omega)^2)$ for any $\boldsymbol{f} \in \mathcal{H}_0^1([0,T];\widetilde{H}^{-1/2}(\mathrm{Div};\Gamma))$ (see [2, Cor. 1]). The continuous dependence of this solution on the data can be inferred from [2, Thm. 2]. From the weak formulation (2.3), it also follows that

$$\mathrm{div}\big(\varepsilon(x)\boldsymbol{E}_{\boldsymbol{f}}(t)\big) \;=\; \mathrm{div}\big(\mu(x)\boldsymbol{H}_{\boldsymbol{f}}(t)\big) \;=\; 0 \qquad \text{for all } t \in [0,T]\,.$$

For a proof of the latter, we can rely on an argument similar to [2, Pro. 1]. □

3 Localized solutions for the acoustic wave equation.

We discuss the existence and construction of localized solutions to the IBVP for the wave equation (2.1). To simplify the presentation, we denote for any subset $D \subseteq \Omega$ or $\Sigma \subseteq \partial\Omega$ and for $0 \le a < b \le T$,

$$D_{a,b} \;:=\; D \times (a,b)\,,\ \Sigma_{a,b} \;:=\; \Sigma \times (a,b) \quad \text{and} \quad D_b \;:=\; D\times(0,b)\,,\ \Sigma_b \;:= \Sigma \times (0,b)\,.$$

Let us also introduce some terminologies regarding the travel time metric associated to wave equation (2.1) and the Maxwell system (2.2). In the latter case the wave speed is given by $c = 1/\sqrt{\varepsilon\mu}$. We denote $\mathrm{d}(x,y)$ as the Riemannian distance

$$\mathrm{d}(x,y) \;:=\; \inf_\gamma \int_\alpha^\beta \frac{|\gamma'(t)|}{c(\gamma(t))}\,\mathrm{d}t\,,$$

between two points $x, y \in \overline{\Omega}$, where the infimum is taken over all smooth curves γ in Ω satisfying $\gamma(\alpha) = x$ and $\gamma(\beta) = y$.

Moreover, let

$$\mathrm{d}(x,\Gamma) \;:=\; \inf_{y\in\Gamma} \mathrm{d}(x,y)\,, \quad x \in \Omega\,, \qquad \text{and} \qquad \mathrm{d}(\Omega,\Gamma) \;:=\; \sup_{x\in\overline{\Omega}} \mathrm{d}(x,\Gamma)\,,$$

and $\mathrm{diam}(\Omega) := \sup_{x,y\in\overline{\Omega}} \mathrm{d}(x,y)$.

We define

$$\mathcal{R}_\tau[g](\,\cdot\,,t) := g(\,\cdot\,,\tau - t) \qquad \text{and} \qquad \mathcal{T}_\tau[g](\,\cdot\,,t) := g(\,\cdot\,,t-\tau)\,, \tag{3.1}$$

which, with respect to the time-level $t = \tau \in \mathbb{R}$, represent the time-reversal and time-translation operator, respectively.

Our analysis of localized waves for the acoustic wave equation relies on a careful investigation of the range spaces of the adjoints of certain restricted solution operators for the IBVP (2.1). For any bounded open subset $B \subset \Omega$ and for $0 \le a < b \le T$ we introduce

$$\mathbb{L}_{B_{a,b}} : L^2(\Gamma_T) \to L^2(B_{a,b})\,, \quad f \mapsto u_f|_{B_{a,b}}\,, \tag{3.2}$$

where u_f denotes the weak solution of (2.1). In the following lemma we identify the adjoint of this operator.

Lemma 3.1. *The adjoint of the operator $\mathbb{L}_{B_{a,b}}$ from (3.2) is given by*

$$\mathbb{L}^*_{B_{a,b}} : L^2(B_{a,b}) \to L^2(\Gamma_T)\,, \quad g \mapsto \partial_{\boldsymbol{\nu}} v_g|_{\Gamma_T}\,, \tag{3.3}$$

where $v_g \in C([0,T]; H^1_0(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ is the unique weak solution of the IBVP

$$c^{-2}(x)\partial_t^2 v_g - \Delta v_g + q(x) v_g = g \qquad \text{in } \Omega_T\,, \tag{3.4a}$$

$$v_g|_{t=T} = \partial_t v_g|_{t=T} = 0 \qquad \text{in } \Omega\,, \tag{3.4b}$$

$$v_g|_{(\partial\Omega)_T} = 0 \qquad \text{on } (\partial\Omega)_T\,. \tag{3.4c}$$

Since this solution satisfies $\partial_{\boldsymbol{\nu}} v_g|_{(\partial\Omega)_T} \in L^2((\partial\Omega)_T)$, the operator $\mathbb{L}^*_{B_{a,b}}$ is well-defined.

Proof. Taking time-reversal into consideration, i.e., considering $\mathcal{R}_T v_g$ instead of v_g and $\mathcal{R}_T g$ instead of g, we can apply [19, Thm. 2.30] to see that, for any $g \in L^2(B_{a,b})$ and after extending this function by zero to all of $\Omega \times (0,T)$, the IBVP (3.4) has a unique weak solution $v_g \in C([0,T]; H^1_0(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ which satisfies the hidden regularity condition $\partial_{\boldsymbol{\nu}} v_g|_{(\partial\Omega)_T} \in L^2((\partial\Omega)_T)$.

Now let $f \in L^2(\Gamma_T)$ and $g \in L^2(B_{a,b})$, and denote by u_f and v_g the corresponding weak solutions of (2.1) and (3.4), respectively. We denote by $\{f_k\}_{k\in\mathbb{N}} \subseteq C_0^\infty([0,T]; C_0^2(\Gamma))$ and by $\{g_k\}_{k\in\mathbb{N}} \subseteq C_0^\infty([0,T]; C_0^\infty(B_{a,b}))$ smooth approximations of f and g, respectively. Then [19, Thm. 2.45] shows that the associated solutions u_{f_k} and v_{g_k} of (2.1) and (3.4) satisfy $u_{f_k} \in C([0,T]; H^2(\Omega)) \cap C^2([0,T]; L^2(\Omega))$ and $v_{g_k} \in C^\infty([0,T]; C^\infty(\Omega))$, respectively. Furthermore, it follows from the continuous dependence of these solutions on the data (see [19, Thm. 2.30]) that $u_f = \lim_{k\to\infty} u_{f_k}$ in $L^2(B_{a,b})$ and $\partial_{\boldsymbol{\nu}} v_g = \lim_{k\to\infty} \partial_{\boldsymbol{\nu}} v_{g_k}$ in $L^2((\partial\Omega)_T)$. Using integration by parts, the strong formulations of (2.1) and (3.4), and a passage to the limit, we observe

$$\begin{aligned}
\langle \mathbb{L}_{B_{a,b}}[f], g \rangle_{L^2(B_{a,b})} &= \int_a^b \int_B u_f g \,\mathrm{d}x \,\mathrm{d}t = \lim_{k\to\infty} \int_a^b \int_B u_{f_k} g_k \,\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{k\to\infty} \int_0^T \int_\Omega \big(c^{-2}\partial_t^2 u_{f_k} - \Delta u_{f_k} + q u_{f_k}\big) v_{g_k} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \lim_{k\to\infty} \int_0^T \int_{\partial\Omega} u_{f_k} \partial_{\boldsymbol{\nu}} v_{g_k} \,\mathrm{d}s_x\,\mathrm{d}t - \lim_{k\to\infty} \int_0^T \int_{\partial\Omega} v_{g_k} \partial_{\boldsymbol{\nu}} u_{f_k} \,\mathrm{d}s_x\,\mathrm{d}t \\
&= \lim_{k\to\infty} \langle f_k, \partial_{\boldsymbol{\nu}} v_{g_k} \rangle_{L^2(\Gamma_T)} = \langle f, \partial_{\boldsymbol{\nu}} v_g \rangle_{L^2(\Gamma_T)}\,.
\end{aligned}$$

This ends the proof. □

3.1 Localization in space

In Theorem 3.2 we establish the existence aspects of solutions to (2.1) that are localized in space. Their construction will be discussed in Corollary 3.4 below.

Theorem 3.2. *Let $D \Subset \Omega$ be open such that $\Omega \setminus \overline{D}$ is connected, and let $B \subset \Omega$ with $B \not\subset D$. For $0 \le a < b \le T$ and $\mathrm{d}(\Omega, \Gamma) < b$, there exists a sequence $\{f_k\}_{k\in\mathbb{N}} \subset L^2(\Gamma_T)$ such that*

$$\|u_{f_k}\|_{L^2(B_{a,b})} \to \infty \qquad \textit{and} \qquad \|u_{f_k}\|_{L^2(D_T)} \to 0 \qquad \textit{as } k \to \infty\,,$$

where u_{f_k} denotes the solution to (2.1) for $f = f_k$.

Proof. We start with a brief sketch of the proof. Note that without loss of generality we can assume that $\overline{B} \cap \overline{D} = \emptyset$ and that $\Omega \setminus (\overline{B \cup D})$ is connected; otherwise we replace B by a sufficiently small open ball $\widetilde{B} \Subset B \setminus \overline{D}$. Defining the two operators $\mathbb{L}_{B_{a,b}} : L^2(\Gamma_T) \to L^2(B_{a,b})$ and $\mathbb{L}_{D_T} : L^2(\Gamma_T) \to L^2(D_T)$ as in (3.2), after replacing $B_{a,b}$ by D_T in this definition for the second one, we may reframe Theorem 3.2 and look for the existence of $\{f_k\}_{k\in\mathbb{N}} \subset L^2(\Gamma_T)$ for which

$$\|\mathbb{L}_{B_{a,b}}[f_k]\|_{L^2(B_{a,b})} \to \infty \qquad \text{and} \qquad \|\mathbb{L}_{D_T}[f_k]\|_{L^2(D_T)} \to 0 \qquad \text{as } k \to \infty\,. \tag{3.5}$$

In view of Lemma A.2 in the appendix, we can alternatively prove the non-inclusion

$$\operatorname{Ran} \mathbb{L}^*_{B_{a,b}} \not\subseteq \operatorname{Ran} \mathbb{L}^*_{D_T}\,, \tag{3.6}$$

in order to establish (3.5). We rely on a contrapositive argument to prove (3.6). For the sake of presentation, we divide the details of our proof in two steps.

Step I: The adjoints of the operators $\mathbb{L}_{B_{a,b}}$ and $\mathbb{L}_{D_T}$ are given by $\mathbb{L}^*_{B_{a,b}} : L^2(B_{a,b}) \to L^2(\Gamma_T)$ and $\mathbb{L}^*_{D_T} : L^2(D_T) \to L^2(\Gamma_T)$ as described in Lemma 3.3, again after replacing $B_{a,b}$ by D_T in this definition for the second one. Let us now assume that

$$h \in \operatorname{Ran} \mathbb{L}^*_{B_{a,b}} \cap \operatorname{Ran} \mathbb{L}^*_{D_T}\,.$$

Then, $h = \partial_{\boldsymbol{\nu}} v_{g_1}|_{\Gamma_T} = \partial_{\boldsymbol{\nu}} v_{g_2}|_{\Gamma_T}$, where v_{g_i}, $i = 1, 2$, solves the IBVP (3.4) for some $g_1 \in L^2(B_{a,b})$ and $g_2 \in L^2(D_T)$, respectively. Denoting $\widetilde{\Omega} := \Omega \setminus (\overline{B \cup D})$ and choosing $\delta > 0$ small enough such that $b - \delta > \max\{\mathrm{d}(\Omega, \Gamma), a\}$, we use a result on the unique continuation of Cauchy data for the wave equation to prove that

$$v_{g_1} \;=\; v_{g_2} \qquad \text{in } \widetilde{\Omega}_{b-\delta,T}\,. \tag{3.7}$$

To see (3.7), we define $w := v_{g_1} - v_{g_2}$ in $\widetilde{\Omega}_T$. We extend w from $\widetilde{\Omega}_T$ to $\widetilde{\Omega}_{2T}$ by zero, and with an abuse of notation we also denote this extension by w. Since $w|_{t=T} = \partial_t w|_{t=T} = 0$ in $\widetilde{\Omega}$, we observe that this extension satisfies $w \in C([0, 2T]; H^1_0(\Omega)) \cap C^1([0, 2T]; L^2(\Omega))$ and

$$\begin{aligned} c^{-2}(x)\partial_t^2 w - \Delta w + q(x)w &= 0 \qquad && \text{in } \widetilde{\Omega}_{2T}\,,\\ w|_{\Gamma_{2T}} = \partial_{\boldsymbol{\nu}} w|_{\Gamma_{2T}} &= 0 && \text{on } \Gamma_{2T}\,. \end{aligned}$$

In view of the vanishing of the Cauchy data of w on Γ_{2T}, the global unique continuation principle for the wave equation [19, Thm 3.16] implies that

$$w \;=\; 0 \qquad \text{in } \left\{(x,t) \in \widetilde{\Omega}_{2T} \;\middle|\; \mathrm{d}(x, \Gamma) \le T - |t - T|\right\}. \tag{3.8}$$

In conjunction with (3.8), the choice of δ gives that $w = 0$ in $\widetilde{\Omega}_{b-\delta,T}$,[2] proving (3.7). Here we used the fact that $\mathrm{d}(\Omega, \Gamma) + \delta < b$.

[2] If $\Gamma = \partial\Omega$, it suffices to assume, $\mathrm{diam}(\Omega) < 2b$.

We now show that

$$v_{g_1} = 0 \quad \text{in } \Omega \setminus \overline{B} \times (b-\delta, T)\,. \tag{3.9}$$

Taking (3.7) into consideration, we can define

$$v_{\rm com} := \begin{cases} v_{g_2} & \text{in } B \times (b-\delta, T)\,, \\ v_{g_1} & \text{in } D \times (b-\delta, T)\,, \\ v_{g_1} = v_{g_2} & \text{in } \widetilde{\Omega}_{b-\delta, T}\,, \end{cases}$$

which satisfies the homogeneous problem

$$\begin{aligned} c^{-2}(x)\partial_t^2 v_{\rm com} - \Delta v_{\rm com} + q(x) v_{\rm com} &= 0 && \text{in } \Omega_{b-\delta,T}\,, \\ v_{\rm com}|_{t=T} = \partial_t v_{\rm com}|_{t=T} &= 0 && \text{in } \Omega\,, \\ v_{\rm com}|_{(\partial\Omega)_{b-\delta,T}} &= 0 && \text{on } (\partial\Omega)_{b-\delta,T}\,. \end{aligned}$$

The above problem, being well-posed, only admits the trivial solution. Therefore, we can conclude that $v_{\rm com} = 0$ in $\Omega_{b-\delta,T}$, which establishes (3.9).

Step II: Next we show that there exists $g \in L^2(B_{a,b})$ such that v_g solving (3.4) satisfies

$$v_g\big|_{(\Omega\setminus\overline{B})\times(b-\delta,T)} \not\equiv 0\,. \tag{3.10}$$

To this end, we employ Lemma 3.3 below for the choice $\tau = \delta$. As a consequence, we have $\widetilde{g} \in L^2(B_\delta)$ such that the solution $w_{\widetilde{g}}$ of (3.11) satisfies $w_{\widetilde{g}} \not\equiv 0$ in $(\Omega \setminus \overline{B}) \times (0, \delta)$. Applying time-translation and time-reversal as introduced in (3.1), we define $g = \mathcal{R}_T \mathcal{T}_{T-b}[\widetilde{g}]$ in Ω_T and notice that $\operatorname{supp} g \subseteq B_{a,b}$. Now let v_g denote the associated solution of (3.4). Then $v_g = 0$ in $\Omega_{b,T}$ and $v_g \not\equiv 0$ in $(\Omega \setminus \overline{B}) \times (b-\delta, b)$, satisfying (3.10). It is immediate from the definition that $\partial_{\boldsymbol{\nu}} v_g\big|_{\Gamma_T} \in \operatorname{Ran} \mathbb{L}^*_{B_{a,b}}$. However, $\partial_{\boldsymbol{\nu}} v_g\big|_{\Gamma_T} \notin \operatorname{Ran} \mathbb{L}^*_{D_T}$. This is because, if $\partial_{\boldsymbol{\nu}} v_g\big|_{\Gamma_T} \in \operatorname{Ran} \mathbb{L}^*_{D_T}$, then we will have $v_g \equiv 0$ in $(\Omega \setminus \overline{B}) \times (b-\delta, T)$ from (3.9) contradicting our choice of g. In conclusion, we have proved (3.6), which ends the proof of Theorem 3.2. $\square$

In Step II of the proof of Theorem 3.2 we have used the following auxiliary result.

Lemma 3.3. *Consider an open subset $B \subset \Omega$ satisfying $\Omega \setminus \overline{B} \neq \emptyset$ and let $0 < \tau \leq T$. There exists a source $g \in L^2(B_\tau)$ for which $w_g|_{(\Omega\setminus\overline{B})\times(0,\tau)} \not\equiv 0$, where $w_g \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ is the unique weak solution of the IBVP*

$$c^{-2}(x)\partial_t^2 w_g - \Delta w_g + q(x) w_g = g \qquad \textit{in } \Omega_T\,, \tag{3.11a}$$

$$w_g|_{t=0} = \partial_t w_g|_{t=0} = 0 \qquad \textit{in } \Omega\,, \tag{3.11b}$$

$$w_g|_{(\partial\Omega)_T} = 0 \qquad \textit{on } (\partial\Omega)_T\,. \tag{3.11c}$$

Moreover, such $g \in L^2(B_\tau)$ can be constructed.

Proof. Denoting $\mathcal{M}^{(\tau)} := \{x \in \Omega \setminus \overline{B} \mid \operatorname{d}(x, \partial B) < \tau\}$, we define the operator

$$\mathbb{T}_\tau : L^2(B_\tau) \to L^2(\mathcal{M}^{(\tau)})\,, \quad g \mapsto w_g|_{\mathcal{M}^{(\tau)} \times \{\tau\}}\,. \tag{3.12}$$

Employing the unique continuation principle for wave equation [19, Thm. 3.16], we can see that $\operatorname{Ran} \mathbb{T}_\tau$ is dense in $L^2(\mathcal{M}^{(\tau)})$. We skip discussing the details here as it follows from adjusting approximate interior controllability for wave equation; see [26, Thm. 3.26]. The amount of time required for such approximate controllability result is 2τ as one aims to control both $w_g|_{t=\tau}$ and $\partial_t w_g|_{t=\tau}$. However we are interested only in $w_g|_{t=\tau}$ and therefore the time-length τ is sufficient for density of the range of $\mathbb{T}_\tau$. This can be proved by means of an odd reflection argument which we showcase in Lemma 4.4 in the context of electrodynamics. In conclusion, we can choose $g \in L^2(B_\tau)$ such that $w_g|_{t=\tau} \not\equiv 0$ in $\mathcal{M}^{(\tau)}$. Note that w_g is continuous in time, implying $w_g|_{(\Omega\setminus\overline{B})\times(0,\tau)} \not\equiv 0$.

Finally we discuss the construction of such a source g. For this, we consider the problem of minimizing the Tikhonov functional

$$\mathbb{J}_\beta[g] := \|\mathbb{T}_\tau[g] - \mathbb{1}_{\mathcal{M}^{(\tau)}}\|^2_{L^2(\mathcal{M}^{(\tau)})} + \beta\|g\|^2_{L^2(B_\tau)}, \qquad \beta > 0, \tag{3.13}$$

where $\mathbb{1}_{\mathcal{M}^{(\tau)}}$ denotes the characteristic function on $\mathcal{M}^{(\tau)}$. If g_β denotes the minimizer for (3.13), then g_β satisfies

$$g_\beta = \left(\mathbb{T}_\tau^*\mathbb{T}_\tau + \beta\mathbb{I}\right)^{-1}\mathbb{T}_\tau^*[\mathbb{1}_{\mathcal{M}^{(\tau)}}], \tag{3.14}$$

and $\lim_{\beta\to 0+}\mathbb{T}_\tau[g_\beta] = \mathbb{1}_{\mathcal{M}^{(\tau)}}$. For a proof of (3.14), we refer the reader, e.g., to [10, p. 52–55]. This shows that g_β given by (3.14) is an example for the desired source g in Lemma 3.3 when $\beta > 0$ is sufficiently small. □

Next we consider the construction of localized waves as in Theorem 3.2.

Corollary 3.4. *A sequence of Dirichlet data $\{f_k\}_{k\in\mathbb{N}} \subset L^2(\Gamma_T)$ as in Theorem 3.2 can be explicitly constructed as*

$$f_k := \frac{1}{\|\sqrt{\mathbb{J}_k}[\xi]\|^{3/2}}\mathbb{J}_k[\xi] \quad \textit{with} \quad \mathbb{J}_k := \left(\mathbb{L}_{D_T}^*\mathbb{L}_{D_T} + k^{-1}\mathbb{I}\right)^{-1}, \qquad k \in \mathbb{N},$$

where, taking $\beta > 0$ sufficiently small, we choose $0 < \tau < \min\{b-a, b-\mathrm{d}(\Omega,\Gamma)\}$ and

$$\xi := \mathbb{L}_{B_{a,b}}^*\Big[\mathcal{R}_T\mathcal{T}_{T-b}\big[(\mathbb{T}_\tau^*\mathbb{T}_\tau + \beta\mathbb{I})^{-1}\mathbb{T}_\tau^*[\mathbb{1}_{\mathcal{M}^{(\tau)}}]\big]\Big].$$

Here, the operators $\mathbb{L}_{B_{a,b}}$, $\mathbb{L}_{D_T}$, and $\mathbb{T}_\tau$ are defined in (3.2) and (3.12), respectively.

Proof. For the construction of localized potentials, we make use of Lemma A.2 with $\mathcal{A}_1 = \mathbb{L}_{B_{a,b}}^*$, $\mathcal{A}_2 = \mathbb{L}_{D_T}^*$, and $\xi = \mathbb{L}_{B_{a,b}}^*[\mathcal{R}_T\mathcal{T}_{T-b}[g]]$ where, for sufficiently small $\beta > 0$, we consider

$$g := \left(\mathbb{T}_\tau^*\mathbb{T}_\tau + \beta\mathbb{I}\right)^{-1}\mathbb{T}_\tau^*[\mathbb{1}_{\mathcal{M}^{(\tau)}}],$$

with $0 < \tau < \min\{b-a, b-\mathrm{d}(\Omega,\Gamma)\}$ and $\mathbb{T}_\tau$ defined in (3.12). The arguments in the Step II of the proof of Theorem 3.2 yield $\xi \notin \operatorname{Ran}\mathbb{L}_{D_T}^*$. If $\mathbb{J}_k := (\mathbb{L}_{D_T}^*\mathbb{L}_{D_T} + k^{-1}\mathbb{I})^{-1}$ then $\mathbb{J}_k$ is positive definite and therefore $\sqrt{\mathbb{J}_k}$ makes sense for $k \in \mathbb{N}$. If we denote $\eta_k := \mathbb{J}_k[\xi]$, then $\langle\xi,\eta_k\rangle = \langle\xi,\mathbb{J}_k[\xi]\rangle = \|\sqrt{\mathbb{J}_k}[\xi]\|^2$ for $k \in \mathbb{N}$. In view of Lemma A.2, the construction of localized wave functions of Theorem 3.2 follows. □

Remark 3.5. Due to the finite speed of propagation, we note that $\mathrm{d}(\Omega,\Gamma)$ is the minimum time needed for the waves to reach every part of Ω. In other words, if $b < \mathrm{d}(\Omega,\Gamma)$, we cannot construct localized waves having arbitrarily large norm on certain open subsets $B \subset \Omega$. This can be justified by a standard domain of dependence argument. See, for instance, [19, Thm. 2.47]. Also notice that the choice of a did not play any role in Theorem 3.2. Therefore we can replace a by a_k in Theorem 3.2 where $0 \le a_k < b$, $k \in \mathbb{N}$ with $a_k \to b$ as $k \to \infty$. This implies that the high energy part of localized waves can correspond to any small time-interval (around b) of our choice whenever the condition $b > \mathrm{d}(\Omega,\Gamma)$ is fulfilled. ◊

Remark 3.6. Theorem 3.2 is also valid for a sequence from any dense subset of $L^2(\Gamma_T)$. This immediately implies that the sequence of Dirichlet data in Theorem 3.2 can be chosen to be C^∞-smooth. As a consequence, one can also improve the smoothness of localized waves. We further add that one can prove results similar to Theorem 3.2 using interior sources (in place of boundary sources) which are supported in ω_T where $\omega \subseteq \Omega$ is an arbitrary open set and $T > 0$ is sufficiently large. ◊

Remark 3.7. Our arguments to construct localized solutions mainly use Tataru's sharp global unique continuation result [32] for wave equation which holds even for more general hyperbolic operators admitting time-independent (or, time-analytic) coefficients such as

$$c^{-2}(x)\partial_t^2 - \operatorname{div}\left(A(x)\nabla_x\right) + B(x)\cdot\nabla_x + q(x)$$

where A and c denote some C^1-smooth positive definite matrix and non-negative functions respectively. Furthermore $B(\cdot)$ and $q(\cdot)$ represent some bounded vector field and scalar function, respectively. Therefore, Theorem 3.2 can also be established for such general operators. ◊

3.2 Localization in time

Similarly we can construct boundary sources for which the solutions to (2.1) concentrate on any given open set $B \subset \Omega$ at a sufficiently large time.

Theorem 3.8. *Consider an open subset $B \Subset \Omega$ such that $\Omega \setminus \overline{B}$ is connected, and let $[a, b]$ and $[c, d]$ be two subintervals of $[0, T]$ with $[a, b] \cap [c, d] = \emptyset$ such that*

(I) $\mathrm{d}(\Omega, \Gamma) < |b - d|$ *if* $c < a$,

(II) $\mathrm{d}(\Omega, \Gamma) < b$ *and* $\mathrm{d}(B, \partial B) < |b - c|/2$ *if* $c > a$.

Then, there exists a sequence $\{f_k\}_{k\in\mathbb{N}} \subset L^2(\Gamma_T)$ such that

$$\|u_{f_k}\|_{L^2(B_{a,b})} \to \infty \qquad \text{and} \qquad \|u_{f_k}\|_{L^2(B_{c,d})} \to 0 \qquad \text{as } k \to \infty\,,$$

where u_{f_k} denotes the solution to (2.1) for $f = f_k$. Moreover, such boundary sources $\{f_k\}_{k\in\mathbb{N}}$ can be constructed.

Proof. We present the proof of Theorem 3.8 in two separate cases.

Case I: ($c < a$) The intervals $[c, d]$ and $[a, b]$ being disjoint, we also have $d < a$. Moreover, it is sufficient to consider f_k belonging to $L^2(\Gamma_b)$. If we can establish the existence of $f \in L^2(\Gamma_{d,b})$ such that the solution u_f to (2.1) satisfies $u_f \not\equiv 0$ in $B_{a,b}$, then we can consider $f_k = kf$ for $k \in \mathbb{N}$ in Γ_T. With such consideration, we have $u_{f_k} = k\, u_f$ in Ω_T and therefore

$$\|u_{f_k}\|_{L^2(B_{a,b})} \to \infty \quad \text{as } k \to \infty \qquad \text{and} \qquad \|u_{f_k}\|_{L^2(B_{c,d})} = 0 \quad \text{for all } k \in \mathbb{N}\,.$$

Now we argue to prove the existence and construction of $f \in L^2(\Gamma_{d,T})$ for which $u_f \not\equiv 0$ in $B_{a,b}$. Note that, $u_f = 0$ in Ω_d. Invoking a time-translation, it therefore reduces to showing $u_f \not\equiv 0$ in $B_{a-d,b-d}$ for some $f \in L^2(\Gamma_{b-d})$. At this point, we take $\tau = b - d$ in Lemma 3.9 below to guarantee the existence of such $f \in L^2(\Gamma_{b-d})$. Here we use that $\mathrm{d}(\Omega, \Gamma) < |b - d|$.

Also, we can consider a minimization argument similar to that of Lemma 3.3 to construct the boundary source

$$f = (\mathbb{P}_\tau^*\mathbb{P}_\tau + \beta\mathbb{I})^{-1}\mathbb{P}_\tau^*[\mathbb{1}_B]\,,$$

with $\mathbb{P}_\tau$ defined in (3.17) below and $\beta > 0$ sufficiently small such that $u_f|_{t=b-d} \not\equiv 0$ in B at time $t = b - d$. Now the continuity of u_f with respect to time ensures that $u_f \not\equiv 0$ in $B_{a-d,b-d}$ completing our argument for Case I.

Case II: ($a < c$) We have $b < c$ as well since $[c, d]$ and $[a, b]$ are disjoint. Here we aim to establish the following relation

$$\operatorname{Ran}\mathbb{L}^*_{B_{a,b}} \cap \operatorname{Ran}\mathbb{L}^*_{B_{c,d}} = \{0\} \tag{3.15}$$

with $\mathbb{L}_{B_{a,b}}$ and $\mathbb{L}_{B_{c,d}}$ defined analogous to (3.2). In view of the relation (3.15), we may refer to Lemma A.2 for the existence of boundary sources f_k needed in Theorem 3.8.

To prove (3.15), we rely on a contrapositive argument as done in Theorem 3.2. Suppose that $h \in \operatorname{Ran}\mathbb{L}^*_{B_{a,b}} \cap \operatorname{Ran}\mathbb{L}^*_{B_{c,d}}$. This implies $h = \partial_{\boldsymbol{\nu}} v_{g_1}|_{\Gamma_T} = \partial_{\boldsymbol{\nu}} v_{g_2}|_{\Gamma_T}$ where v_{g_1} and v_{g_2} solve the adjoint problem (3.4) corresponding to sources $g_1 \in L^2(B_{a,b})$ and $g_2 \in L^2(B_{c,d})$, respectively. Since $g_1 = g_2 = 0$ in $B_{d,T}$, we obtain from (3.4) that $v_{g_1} = v_{g_2} = 0$ in $\Omega_{d,T}$. In particular, $h|_{\Gamma_{d,T}} = 0$. We denote $\widetilde{\Omega} := \Omega \setminus \overline{B}$ and define $w := v_{g_1} - v_{g_2}$ in Ω_d. Extending w from Ω_d to Ω_{2d} by zero, and denoting this extension again by w, we find that $w \in C([0, 2d]; H^1_0(\Omega)) \cap C^1([0, 2d]; L^2(\Omega))$ and

$$\begin{aligned} c^{-2}(x)\partial_t^2 w - \Delta w + q(x)w &= 0 && \text{in } \widetilde{\Omega}_{2d}\,, \\ w|_{t=d} = \partial_t w|_{t=d} &= 0 && \text{in } \Omega\,, \\ w|_{\Gamma_{2d}} = \partial_{\boldsymbol{\nu}} w|_{\Gamma_{2d}} &= 0 && \text{on } \Gamma_{2d}\,. \end{aligned}$$

Invoking the unique continuation argument [19, Thm. 3.16], we see that

$$w = 0 \qquad \text{in } \left\{(x,t) \in \widetilde{\Omega}_{2d} \,\middle|\, \mathrm{d}(x,\Gamma) \le d - |t-d|\right\}.$$

In view of our assumption $\mathrm{d}(\Omega,\Gamma) < b$, we obtain $w(x,t) = 0$, i.e., $v_{g_1} = v_{g_2}$ in $\widetilde{\Omega}_{b,d}$. Since $g_1(\cdot,t) = 0$ for almost every $t \in (b,T)$, we also have $v_{g_1} = 0$ in $\Omega_{b,d}$. In particular, $h|_{\Gamma_{b,d}} = 0$. Furthermore, this implies $v_{g_2} = 0$ in $\widetilde{\Omega}_{b,c}$. Again we use the unique continuation argument [19, Thm. 3.16] to see that

$$v_{g_2} = 0 \qquad \text{in } \left\{(x,t) \in B_{b,c} \,\middle|\, \mathrm{d}(x,\partial B) < \frac{c-b}{2} + \left|t - \frac{c+b}{2}\right|\right\},$$

which, due to the assumption $\mathrm{d}(B,\partial B) < |b-c|/2$, yields $v_{g_2} = \partial_t v_{g_2} = 0$ in the slice $B \times \{\frac{c+b}{2}\}$. Together with what we have seen before, we conclude that $v_{g_2} = \partial_t v_{g_2} = 0$ on $\Omega \times \{\frac{c+b}{2}\}$. Since $g_2(\cdot,t) = 0$ for almost every $t \in (0, \frac{c+b}{2})$, we use the uniqueness of solutions to the adjoint problem to conclude $v_{g_2} = 0$ in $\Omega_{\frac{c+b}{2}}$ which further gives $h = 0$ on Γ_b. Combining this with $h|_{\Gamma_{b,d}} = 0$ and $h|_{\Gamma_{d,T}} = 0$, our proof for (3.15) is complete.

For the construction of f_k, we need to first obtain any non-zero element from $\operatorname{Ran}\mathbb{L}^*_{B_{a,b}}$, say ξ, and then appeal to Lemma A.2 with the choice

$$f_k := \frac{1}{\|\sqrt{\mathbb{J}_k[\xi]}\|^{3/2}}\mathbb{J}_k[\xi] \quad \text{with} \quad \mathbb{J}_k := \left(\mathbb{L}^*_{B_{c,d}}\mathbb{L}_{B_{c,d}} + k^{-1}\mathbb{I}\right)^{-1}, \qquad k \in \mathbb{N}.$$

Now, to find $0 \ne \xi \in \operatorname{Ran}\mathbb{L}^*_{B_{a,b}}$, we may argue as in Step II of Theorem 3.2, which in turn relies on Lemma 3.3. For brevity, we briefly discuss the steps. We first choose δ satisfying $\mathrm{d}(\Omega,\Gamma)+\delta < b$, then we consider

$$g := \left(\mathbb{T}^*_\delta\mathbb{T}_\delta + \beta\mathbb{I}\right)^{-1}\mathbb{T}^*_\delta[\mathbb{1}_{\mathcal{M}^{(\delta)}}]$$

for $\beta > 0$ sufficiently small, where $\mathbb{T}_\delta$ is defined in (3.12) for $\tau = \delta$. We observe that $\xi \not\equiv 0$ in Γ_T where

$$\xi := \mathbb{L}^*_{B_{a,b}}\left[\mathcal{R}_T\mathcal{T}_{T-b}[g]\right] \tag{3.16}$$

If $\xi \equiv 0$ on Γ_T, then we can again use unique continuation (see [19, Thm. 3.16]) to claim that $v_{\mathcal{R}_T \mathcal{T}_{T-b}[g]}$ solving (3.4) satisfies the vanishing condition

$$v_{\mathcal{R}_T \mathcal{T}_{T-b}[g]}(x,t) = 0 \qquad \text{in } (\Omega \setminus \overline{B}) \times (b-\delta, b)\,,$$

which contradicts our choice of g implying ξ defined in (3.16) does not vanish on all of Γ_T. □

In Case I of the proof of Theorem 3.8 we used the following auxiliary result.

Lemma 3.9. *For $\tau > \mathrm{d}(\Omega, \Gamma)$ and $B \subseteq \Omega$ open, we can construct $f \in L^2(\Gamma_\tau)$ such that $u_f|_{t=\tau} \not\equiv 0$ in B, where u_f denotes the solution to (2.1).*

We omit the proof of Lemma 3.9, as the arguments are similar to that of Lemma 3.3. It relies on minimizing a Tikhonov functional along with a density result which implies $\overline{\mathrm{Ran}\,\mathbb{P}_\tau} = L^2(\Omega)$, with $\mathbb{P}_\tau$ denoting the mapping

$$\mathbb{P}_\tau : L^2(\Gamma_\tau) \to L^2(\Omega)\,, \quad f \mapsto u_f|_{t=\tau}\,, \tag{3.17}$$

and u_f solving the IBVP (2.1). Once again, we underline to the point that the density of the range of $\mathbb{P}_\tau$ requires adjusting the standard boundary approximate controllability result for the wave equation given by, for instance, [19, Thm. 4.28]. In this regard, we can use an odd reflection argument as discussed in Lemma 4.4 below.

4 Localized solutions for the Maxwell equations.

Next we discuss the existence and construction of localized solutions of the IBVP for Maxwell's equations (2.2). As in Section 3, our proofs rely on an analysis of the range spaces of the adjoints of certain restricted solution operators. We start by introducing these operators and identifying their adjoints.

For any bounded open subset $B \subset \Omega$ and $0 \le a < b \le T$, we define

$$\mathbb{L}_{B_{a,b}} : \mathcal{H}_0^1([0,T]; \widetilde{H}^{-1/2}(\mathrm{Div};\Gamma)) \to \boldsymbol{L}^2(B_{a,b})^2\,, \qquad \boldsymbol{f} \mapsto \left(\boldsymbol{E_f}\big|_{B_{a,b}}, \boldsymbol{H_f}\big|_{B_{a,b}}\right), \tag{4.1a}$$

$$\mathbb{E}_{B_{a,b}} : \mathcal{H}_0^1([0,T]; \widetilde{H}^{-1/2}(\mathrm{Div};\Gamma)) \to \boldsymbol{L}^2(B_{a,b})\,, \qquad \boldsymbol{f} \mapsto \boldsymbol{E_f}\big|_{B_{a,b}}\,, \tag{4.1b}$$

$$\mathbb{H}_{B_{a,b}} : \mathcal{H}_0^1([0,T]; \widetilde{H}^{-1/2}(\mathrm{Div};\Gamma)) \to \boldsymbol{L}^2(B_{a,b})\,, \qquad \boldsymbol{f} \mapsto \boldsymbol{H_f}\big|_{B_{a,b}}\,, \tag{4.1c}$$

where $(\boldsymbol{E_f}, \boldsymbol{H_f})$ denotes the weak solution of (2.2). In the following lemma we identify the adjoints of the three operators from (4.1). Here, we consider an inner product in $\boldsymbol{L}^2(B_{a,b})^2$, which is equivalent to the standard inner product. For $\boldsymbol{J} = (\boldsymbol{j}_1, \boldsymbol{j}_2), \boldsymbol{K} = (\boldsymbol{k}_1, \boldsymbol{k}_2) \in \boldsymbol{L}^2(B_{a,b})^2$, we define

$$\langle \boldsymbol{J}, \boldsymbol{K} \rangle_{\varepsilon,\mu} := \int_{B_{a,b}} \varepsilon(x) \boldsymbol{j}_1(x,t) \cdot \boldsymbol{k}_1(x,t)\, \mathrm{d}x\, \mathrm{d}t + \int_{B_{a,b}} \mu(x) \boldsymbol{j}_2(x,t) \cdot \boldsymbol{k}_2(x,t)\, \mathrm{d}x\, \mathrm{d}t$$

We further introduce two equivalent inner products for $\boldsymbol{L}^2(B_{a,b})$ given by

$$\langle \boldsymbol{j}, \boldsymbol{k} \rangle_\varepsilon := \int_{B_{a,b}} \varepsilon(x) \boldsymbol{j}(x,t) \cdot \boldsymbol{k}(x,t)\, \mathrm{d}x\, \mathrm{d}t\,, \quad \langle \boldsymbol{j}, \boldsymbol{k} \rangle_\mu := \int_{B_{a,b}} \mu(x) \boldsymbol{j}(x,t) \cdot \boldsymbol{k}(x,t)\, \mathrm{d}x\, \mathrm{d}t \tag{4.2}$$

for $\boldsymbol{j}, \boldsymbol{k} \in \boldsymbol{L}^2(B_{a,b})$. The inner products $\langle \cdot, \cdot \rangle_\varepsilon$ and $\langle \cdot, \cdot \rangle_\mu$ are used for the definition of $\mathbb{E}^*_{B_{a,b}}$ and $\mathbb{H}^*_{B_{a,b}}$, respectively. In addition to (3.1), we introduce the notation

$$\mathcal{S}_\tau[g](\,\cdot\,, t) := \int_\tau^t g(\,\cdot\,, s)\, \mathrm{d}s\,,$$

which, with respect to the time-level $t = \tau \in \mathbb{R}$, represents the time-integral operator.

Lemma 4.1. *The adjoints of the operators* $\mathbb{L}_{B_{a,b}}$, $\mathbb{E}_{B_{a,b}}$, *and* $\mathbb{H}_{B_{a,b}}$ *from* (4.1) *are given by*

$$\mathbb{L}^*_{B_{a,b}} : \boldsymbol{L}^2(B_{a,b})^2 \to \mathcal{H}^1_0([0,T]; \widetilde{H}^{-1/2}(\mathrm{Div};\Gamma)^*), \quad \boldsymbol{J} \mapsto -\mathcal{S}_0\big[\pi_\tau[\widetilde{\boldsymbol{H}}_{\boldsymbol{J}}]\big|_{\Gamma_T}\big], \tag{4.3a}$$

$$\mathbb{E}^*_{B_{a,b}} : \boldsymbol{L}^2(B_{a,b}) \to \mathcal{H}^1_0([0,T]; \widetilde{H}^{-1/2}(\mathrm{Div};\Gamma)^*), \quad \boldsymbol{j}_1 \mapsto -\mathcal{S}_0\big[\pi_\tau[\widetilde{\boldsymbol{H}}_{(\boldsymbol{j}_1,0)}]\big|_{\Gamma_T}\big], \tag{4.3b}$$

$$\mathbb{H}^*_{B_{a,b}} : \boldsymbol{L}^2(B_{a,b}) \to \mathcal{H}^1_0([0,T]; \widetilde{H}^{-1/2}(\mathrm{Div};\Gamma)^*), \quad \boldsymbol{j}_2 \mapsto -\mathcal{S}_0\big[\pi_\tau[\widetilde{\boldsymbol{H}}_{(0,\boldsymbol{j}_2)}]\big|_{\Gamma_T}\big], \tag{4.3c}$$

where $(\widetilde{\boldsymbol{E}}_{\boldsymbol{J}}, \widetilde{\boldsymbol{H}}_{\boldsymbol{J}}) \in C([0,T]; \boldsymbol{H}_0(\mathbf{curl};\Omega) \times \boldsymbol{H}(\mathbf{curl};\Omega)) \cap C^1([0,T]; \boldsymbol{L}^2(\Omega)^2)$ *denotes, for any* $\boldsymbol{J} = (\boldsymbol{j}_1, \boldsymbol{j}_2) \in \boldsymbol{L}^2(B_{a,b})^2$, *the unique weak solution of the IBVP*

$$\varepsilon(x)\partial_t\widetilde{\boldsymbol{E}}_{\boldsymbol{J}} - \mathbf{curl}\,\widetilde{\boldsymbol{H}}_{\boldsymbol{J}} = \varepsilon(x)\mathcal{S}_T[\boldsymbol{j}_1] \qquad \text{in } \Omega_T, \tag{4.4a}$$

$$\mu(x)\partial_t\widetilde{\boldsymbol{H}}_{\boldsymbol{J}} + \mathbf{curl}\,\widetilde{\boldsymbol{E}}_{\boldsymbol{J}} = \mu(x)\mathcal{S}_T[\boldsymbol{j}_2] \qquad \text{in } \Omega_T, \tag{4.4b}$$

$$\widetilde{\boldsymbol{E}}_{\boldsymbol{J}}|_{t=T} = \widetilde{\boldsymbol{H}}_{\boldsymbol{J}}|_{t=T} = 0 \qquad \text{in } \Omega, \tag{4.4c}$$

$$\boldsymbol{\nu} \times \widetilde{\boldsymbol{E}}_{\boldsymbol{J}}|_{(\partial\Omega)_T} = 0 \qquad \text{on } (\partial\Omega)_T. \tag{4.4d}$$

Proof. To study the well-posedness of (4.4) we consider a time-reversal to convert the backward problem into a forward problem. Then, observing that the source terms $\varepsilon(x)\mathcal{S}_T[\boldsymbol{j}_1]$ and $\mu(x)\mathcal{S}_T[\boldsymbol{j}_2]$ in (4.4) belong to $H^1((0,T); L^2(\Omega))$, we can apply [22, Thm. 5.3] to see that the IBVP (4.4) has a unique weak solution $(\widetilde{\boldsymbol{E}}_{\boldsymbol{J}}, \widetilde{\boldsymbol{H}}_{\boldsymbol{J}}) \in C([0,T]; \boldsymbol{H}_0(\mathbf{curl};\Omega) \times \boldsymbol{H}(\mathbf{curl};\Omega)) \cap C^1([0,T]; \boldsymbol{L}^2(\Omega)^2)$.

Next we discuss the characterization of the adjoint operator $\mathbb{L}^*_{B_{a,b}}$ and note that the characterizations of $\mathbb{E}^*_{B_{a,b}}$ and $\mathbb{H}^*_{B_{a,b}}$ can be proved similarly. Considering $\boldsymbol{f} \in \mathcal{H}^1_0([0,T]; \widetilde{H}^{-1/2}(\mathrm{Div};\Gamma))$ and $(\boldsymbol{j}_1, \boldsymbol{j}_2) = \boldsymbol{J} \in \boldsymbol{L}^2(B_{a,b})^2$, we denote the associated weak solutions to (2.2) and (4.4) by $(\boldsymbol{E}_{\boldsymbol{f}}, \boldsymbol{H}_{\boldsymbol{f}})$ and $(\widetilde{\boldsymbol{E}}_{\boldsymbol{J}}, \widetilde{\boldsymbol{H}}_{\boldsymbol{J}})$, respectively. Recalling from Proposition 2.2 that $(\widetilde{\boldsymbol{E}}_{\boldsymbol{J}}, \widetilde{\boldsymbol{H}}_{\boldsymbol{J}})$ is just continuous in time, we choose smooth approximations $\{\boldsymbol{f}_k\}_{k\in\mathbb{N}} \subset C^\infty_c((0,T]; \widetilde{H}^{-1/2}(\mathrm{Div};\Gamma))$ such that $\boldsymbol{f}_k \to \boldsymbol{f}$ in $\mathcal{H}^1_0([0,T]; \widetilde{H}^{-1/2}(\mathrm{Div};\Gamma))$. Accordingly, we denote by $(\boldsymbol{E}_{\boldsymbol{f}_k}, \boldsymbol{H}_{\boldsymbol{f}_k})$ the associated weak solution to (2.2). The continuous dependence of solutions to (2.2) on the data (see [2, Thm. 2]) shows that $(\boldsymbol{E}_{\boldsymbol{f}_k}, \boldsymbol{H}_{\boldsymbol{f}_k}) \to (\boldsymbol{E}_{\boldsymbol{f}}, \boldsymbol{H}_{\boldsymbol{f}})$ in $C([0,T]; \boldsymbol{L}^2(\Omega)^2)$. Furthermore, we have $(\boldsymbol{E}_{\boldsymbol{f}_k}, \boldsymbol{H}_{\boldsymbol{f}_k}) \in C^\infty([0,T]; \boldsymbol{H}_0(\mathbf{curl};\Omega) \times \boldsymbol{H}(\mathbf{curl};\Omega))$ (see [22, Thm. 2.6]) along with the relations $\partial_t^p \boldsymbol{E}_{\boldsymbol{f}_k} = \boldsymbol{E}_{\partial_t^p \boldsymbol{f}_k}$ and $\partial_t^p \boldsymbol{H}_{\boldsymbol{f}_k} = \boldsymbol{H}_{\partial_t^p \boldsymbol{f}_k}$ for any $k, p \in \mathbb{N}$. Here the weak solutions to (2.2) corresponding to boundary data $\partial_t^p \boldsymbol{f}_k$ are denoted by $(\boldsymbol{E}_{\partial_t^p \boldsymbol{f}_k}, \boldsymbol{H}_{\partial_t^p \boldsymbol{f}_k})$.

Now we use an integration by parts to compute

$$\begin{aligned}
\big\langle \mathbb{L}_{B_{a,b}}[\boldsymbol{f}], \boldsymbol{J} \big\rangle_{\varepsilon,\mu} &= \int_a^b \int_B \big(\varepsilon(x)\boldsymbol{E}_{\boldsymbol{f}} \cdot \boldsymbol{j}_1 + \mu(x)\boldsymbol{H}_{\boldsymbol{f}} \cdot \boldsymbol{j}_2\big) \,\mathrm{d}x \,\mathrm{d}t \\
&= \lim_{k\to\infty} \int_0^T \int_\Omega \big(\varepsilon(x)\boldsymbol{E}_{\boldsymbol{f}_k} \cdot \boldsymbol{j}_1 + \mu(x)\boldsymbol{H}_{\boldsymbol{f}_k} \cdot \boldsymbol{j}_2\big) \,\mathrm{d}x \,\mathrm{d}t \\
&= -\lim_{k\to\infty} \int_0^T \int_\Omega \big(\boldsymbol{E}_{\boldsymbol{f}_k'} \cdot (\varepsilon(x)\mathcal{S}_T[\boldsymbol{j}_1]) + \boldsymbol{H}_{\boldsymbol{f}_k'} \cdot (\mu(x)\mathcal{S}_T[\boldsymbol{j}_2])\big) \,\mathrm{d}x \,\mathrm{d}t \\
&= -\lim_{k\to\infty} \int_0^T \int_\Omega \Big(\boldsymbol{E}_{\boldsymbol{f}_k'} \cdot \big(\varepsilon(x)\partial_t\widetilde{\boldsymbol{E}}_{\boldsymbol{J}} - \mathbf{curl}\,\widetilde{\boldsymbol{H}}_{\boldsymbol{J}}\big) + \boldsymbol{H}_{\boldsymbol{f}_k'} \cdot \big(\mu(x)\partial_t\widetilde{\boldsymbol{H}}_{\boldsymbol{J}} + \mathbf{curl}\,\widetilde{\boldsymbol{E}}_{\boldsymbol{J}}\big)\Big) \,\mathrm{d}x \,\mathrm{d}t \\
&= \lim_{k\to\infty} \int_0^T \int_\Omega \Big(\widetilde{\boldsymbol{E}}_{\boldsymbol{J}} \cdot \big(\varepsilon(x)\partial_t\boldsymbol{E}_{\boldsymbol{f}_k'} - \mathbf{curl}\,\boldsymbol{H}_{\boldsymbol{f}_k'}\big) + \widetilde{\boldsymbol{H}}_{\boldsymbol{J}} \cdot \big(\mu(x)\partial_t\boldsymbol{H}_{\boldsymbol{f}_k'} + \mathbf{curl}\,\boldsymbol{E}_{\boldsymbol{f}_k'}\big)\Big) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad - \lim_{k\to\infty} \int_0^T \int_{\partial\Omega} \gamma_\tau[\boldsymbol{E}_{\boldsymbol{f}_k'}] \cdot \pi_\tau[\widetilde{\boldsymbol{H}}_{\boldsymbol{J}}] \,\mathrm{d}s_x \,\mathrm{d}t \\
&= -\lim_{k\to\infty} \int_0^T \int_\Gamma \boldsymbol{f}_k' \cdot \pi_\tau[\widetilde{\boldsymbol{H}}_{\boldsymbol{J}}] \,\mathrm{d}s_x \,\mathrm{d}t = -\int_0^T \int_\Gamma \big\langle \partial_t \boldsymbol{f}, \partial_t\mathcal{S}_0\big[\pi_\tau[\widetilde{\boldsymbol{H}}_{\boldsymbol{J}}]\big]\big\rangle \,\mathrm{d}s_x \,\mathrm{d}t
\end{aligned}$$

$$= \big\langle \boldsymbol{f}, -\mathcal{S}_0\big[\pi_\tau[\widetilde{\boldsymbol{H}}_{\boldsymbol{J}}]\big]\big\rangle_{\mathcal{H}_0^1([0,T];\widetilde{H}^{-1/2}(\mathrm{Div};\Gamma))}\,,$$

which gives the characterization of $\mathbb{L}^*_{B_{a,b}}$. □

The reason for not using the standard inner product on $\boldsymbol{L}^2(B_{a,b})^2$ in the definition of the adjoint operator in Lemma 4.1 stems from the following observation, which will be used later in the proof of Theorem 4.6.

Remark 4.2. We recall the Helmholtz decompositions

$$\boldsymbol{L}^2(\Omega) \;=\; \{\boldsymbol{V} \in \boldsymbol{L}^2(\Omega) \mid \mathrm{div}(\varepsilon \boldsymbol{V}) = 0\} \oplus \nabla H_0^1(\Omega) \;=\; \{\boldsymbol{V} \in \boldsymbol{L}^2(\Omega) \mid \mathrm{div}(\mu \boldsymbol{V}) = 0\} \oplus \nabla H_0^1(\Omega)$$

(see, e.g., [21, Thm. 4.23]), where the divergence has to be understood in weak sense. These are orthogonal with respect to the inner products $\langle\cdot,\cdot\rangle_\varepsilon$ and $\langle\cdot,\cdot\rangle_\mu$, respectively. Accordingly, we can decompose $\boldsymbol{J} = (\boldsymbol{j}_1, \boldsymbol{j}_1) \in \boldsymbol{L}^2(B_{a,b})^2$ into

$$\begin{aligned}
\boldsymbol{j}_1 &= \widetilde{\boldsymbol{j}}_1 + \nabla\phi_1 \qquad &&\text{with } \mathrm{div}\big(\varepsilon(x)\widetilde{\boldsymbol{j}}_1(t)\big) = 0 \text{ in } \Omega \text{ and } \phi_1(t) \in H_0^1(\Omega)\,,\\
\boldsymbol{j}_2 &= \widetilde{\boldsymbol{j}}_2 + \nabla\phi_2 \qquad &&\text{with } \mathrm{div}\big(\mu(x)\widetilde{\boldsymbol{j}}_2(t)\big) = 0 \text{ in } \Omega \text{ and } \phi_2(t) \in H_0^1(\Omega)\,,
\end{aligned}$$

for a.e. $t \in (a,b)$. Denoting $\widetilde{\boldsymbol{J}} := (\widetilde{\boldsymbol{j}}_1, \widetilde{\boldsymbol{j}}_2)$ and using the fact that the solution $(\boldsymbol{E}_{\boldsymbol{f}}, \boldsymbol{H}_{\boldsymbol{f}})$ to (2.2) satisfies

$$\mathrm{div}(\varepsilon(x)\boldsymbol{E}_{\boldsymbol{f}}(t)) \;=\; \mathrm{div}(\mu(x)\boldsymbol{H}_{\boldsymbol{f}}(t)) \;=\; 0 \qquad \text{for all } t \in [0,T]\,,$$

we obtain that $\mathbb{L}^*_{B_{a,b}}[\boldsymbol{J}] = \mathbb{L}^*_{\Omega_{a,b}}[\widetilde{\boldsymbol{J}}]$. We can draw similar conclusions for $\mathbb{E}^*_{B_{a,b}}$ and $\mathbb{H}^*_{B_{a,b}}$. ◊

4.1 Localization in space

In Theorem 3.2 we establish the existence aspects of solutions to (2.2) that are localized in space. Their construction will be discussed in Corollary 4.5 below. Since the unique continuation principles from [6, 7] require C^2-smooth electric permittivity ε and magnetic permeability μ, we add this assumption in the following theorems.

Theorem 4.3. *Suppose, in addition to our previous assumptions, that $\varepsilon,\mu \in C^2(\overline{\Omega})$. Let $D \Subset \Omega$ be open such that $\Omega\setminus\overline{D}$ is connected, and let $B \subset \Omega$ with $B \not\subset D$. For $0 \le a < b \le T$ and $\mathrm{d}(\Omega,\Gamma) < b$ there exists a sequence $\{\boldsymbol{f}_k\}_{k\in\mathbb{N}} \subset \mathcal{H}_0^1([0,T];\widetilde{H}^{-1/2}(\mathrm{Div};\Gamma))$ such that*

$$\|\boldsymbol{E}_{\boldsymbol{f}_k}\|_{\boldsymbol{L}^2(B_{a,b})} \to \infty \qquad \text{and} \qquad \|\boldsymbol{E}_{\boldsymbol{f}_k}\|_{\boldsymbol{L}^2(D_T)} + \|\boldsymbol{H}_{\boldsymbol{f}_k}\|_{\boldsymbol{L}^2(D_T)} \to 0 \qquad \text{as } k\to\infty\,, \tag{4.5}$$

where $(\boldsymbol{E}_{\boldsymbol{f}_k}, \boldsymbol{H}_{\boldsymbol{f}_k})$ denotes the solution to (2.2) for $\boldsymbol{f} = \boldsymbol{f}_k$.

Furthermore, there exists a sequence $\widetilde{\boldsymbol{f}}_k \in \mathcal{H}_0^1([0,T];\widetilde{H}^{-1/2}(\mathrm{Div};\Gamma))$ such that

$$\|\boldsymbol{H}_{\widetilde{\boldsymbol{f}}_k}\|_{\boldsymbol{L}^2(B_{a,b})} \to \infty \qquad \text{and} \qquad \|\boldsymbol{E}_{\widetilde{\boldsymbol{f}}_k}\|_{\boldsymbol{L}^2(D_T)} + \|\boldsymbol{H}_{\widetilde{\boldsymbol{f}}_k}\|_{\boldsymbol{L}^2(D_T)} \to 0 \qquad \text{as } k\to\infty\,, \tag{4.6}$$

where $(\boldsymbol{E}_{\widetilde{\boldsymbol{f}}_k}, \boldsymbol{H}_{\widetilde{\boldsymbol{f}}_k})$ denotes the solution to (2.2) for $\boldsymbol{f} = \widetilde{\boldsymbol{f}}_k$.

Proof. The arguments that we use to establish Theorem 4.3 will be similar to those in the proof of Theorem 3.2. Therefore, we discuss the important steps only. Also, we limit ourselves to the construction of $\{\boldsymbol{f}_k\}_{k\in\mathbb{N}}$. For the discussion on $\{\widetilde{\boldsymbol{f}}_k\}_{k\in\mathbb{N}}$ one can replace the operator $\mathbb{E}_{B_{a,b}}$ with $\mathbb{H}_{B_{a,b}}$ in the following arguments and adjust Lemma 4.4 appropriately.

Step I: We can again assume without loss of generality that $\overline{B}\cap\overline{D} = \emptyset$ and that $\Omega\setminus\overline{B\cup D}$ is connected. Defining the two operators $\mathbb{E}_{B_{a,b}} : \mathcal{H}_0^1([0,T];\widetilde{H}^{-1/2}(\mathrm{Div};\Gamma)) \to \boldsymbol{L}^2(B_{a,b})$ and

$\mathbb{L}_{D_T} : \mathcal{H}_0^1([0,T]; \widetilde{H}^{-1/2}(\mathrm{Div};\Gamma)) \to \boldsymbol{L}^2(D_T)^2$ as in (4.1), after replacing $B_{a,b}$ by D_T for the second one, we prove by contraposition that

$$\operatorname{Ran} \mathbb{E}^*_{B_{a,b}} \not\subseteq \operatorname{Ran} \mathbb{L}^*_{D_T} . \tag{4.7}$$

To see (4.7), we assume that $\boldsymbol{h} \in \operatorname{Ran} \mathbb{E}^*_{B_{a,b}} \cap \operatorname{Ran} \mathbb{L}^*_{D_T}$, i.e.,

$$\boldsymbol{h} \;=\; -\mathcal{S}_0\big[\pi_\tau[\widetilde{\boldsymbol{H}}_{(\boldsymbol{j}_1,0)}]\big|_{\Gamma_T}\big] \;=\; -\mathcal{S}_0\big[\pi_\tau[\widetilde{\boldsymbol{H}}_{\boldsymbol{K}}]\big|_{\Gamma_T}\big] \,,$$

where $(\widetilde{\boldsymbol{E}}_{(\boldsymbol{j}_1,0)}, \widetilde{\boldsymbol{H}}_{(\boldsymbol{j}_1,0)})$ and $(\widetilde{\boldsymbol{E}}_{\boldsymbol{K}}, \widetilde{\boldsymbol{H}}_{\boldsymbol{K}})$ solve the IBVP (4.4) with source terms $(\varepsilon(x)\mathcal{S}_T[\boldsymbol{j}_1], 0)$ and $(\varepsilon(x)\mathcal{S}_T[\boldsymbol{k}_1], \mu(x)\mathcal{S}_T[\boldsymbol{k}_2])$ for some $\boldsymbol{j}_1 \in \boldsymbol{L}^2(B_{a,b})$ and $\boldsymbol{K} = (\boldsymbol{k}_1, \boldsymbol{k}_2) \in \boldsymbol{L}^2(D_T)^2$, respectively. Next, we define $\widetilde{\Omega} := \Omega \setminus \overline{(B \cup D)}$ and choose $\delta > 0$ such that $b - \delta > \max\{\mathrm{d}(\Omega, \Gamma), a\}$. Using a unique continuation argument, we will show that

$$\widetilde{\boldsymbol{E}}_{(\boldsymbol{j}_1,0)} \;=\; \widetilde{\boldsymbol{E}}_{\boldsymbol{K}} \qquad \text{and} \qquad \widetilde{\boldsymbol{H}}_{(\boldsymbol{j}_1,0)} \;=\; \widetilde{\boldsymbol{H}}_{\boldsymbol{K}} \qquad \text{in } \widetilde{\Omega}_{b-\delta,T} \,. \tag{4.8}$$

To this end, we denote $\widetilde{\boldsymbol{E}} := \widetilde{\boldsymbol{E}}_{(\boldsymbol{j}_1,0)} - \widetilde{\boldsymbol{E}}_{\boldsymbol{K}}$ and $\widetilde{\boldsymbol{H}} := \widetilde{\boldsymbol{H}}_{(\boldsymbol{j}_1,0)} - \widetilde{\boldsymbol{H}}_{\boldsymbol{K}}$ in Ω_T and observe that

$$\mathcal{S}_0\big[\pi_\tau[\widetilde{\boldsymbol{H}}]\big|_{\Gamma_T}\big] \;=\; 0 \,,$$

which implies that $\pi_\tau[\widetilde{\boldsymbol{H}}] = 0$ on Γ_T. Next, we extend $\widetilde{\boldsymbol{E}}$ and $\widetilde{\boldsymbol{H}}$ from $\widetilde{\Omega}_T$ to $\widetilde{\Omega}_{2T}$ by zero, and denote these extensions again by $\widetilde{\boldsymbol{E}}$ and $\widetilde{\boldsymbol{H}}$, respectively. Since $\widetilde{\boldsymbol{E}}|_{t=T} = \widetilde{\boldsymbol{H}}|_{t=T} = 0$ in $\widetilde{\Omega}$, we observe that $(\widetilde{\boldsymbol{E}}, \widetilde{\boldsymbol{H}}) \in C([0,2T]; \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\Omega}) \times \boldsymbol{H}(\mathbf{curl}; \widetilde{\Omega})) \cap C^1([0,2T]; \boldsymbol{L}^2(\widetilde{\Omega})^2)$ and

$$\begin{aligned}
\varepsilon(x)\partial_t\widetilde{\boldsymbol{E}} - \mathbf{curl}\,\widetilde{\boldsymbol{H}} &= 0 && \text{in } \widetilde{\Omega}_{2T} \,,\\
\mu(x)\partial_t\widetilde{\boldsymbol{H}} + \mathbf{curl}\,\widetilde{\boldsymbol{E}} &= 0 && \text{in } \widetilde{\Omega}_{2T} \,,\\
\boldsymbol{\nu} \times \widetilde{\boldsymbol{E}}\big|_{\Gamma_{2T}} = \boldsymbol{\nu} \times \widetilde{\boldsymbol{H}}\big|_{\Gamma_{2T}} &= 0 && \text{on } \Gamma_{2T} \,.
\end{aligned}$$

Furthermore, we obtain from the vanishing of $\widetilde{\boldsymbol{E}}|_{t=T}$ and $\widetilde{\boldsymbol{H}}|_{t=T}$ that

$$\operatorname{div}(\varepsilon\widetilde{\boldsymbol{E}}) \;=\; \operatorname{div}(\mu\widetilde{\boldsymbol{H}}) \;=\; 0 \qquad \text{in } \widetilde{\Omega}_{2T} \,.$$

Applying the unique continuation principle for the Maxwell equations from [6, Thm. 4.5] (see also [7, Cor. 1.2] for the global version), we have

$$\widetilde{\boldsymbol{E}} \;=\; \widetilde{\boldsymbol{H}} \;=\; 0 \qquad \text{in } \big\{(x,t) \in \widetilde{\Omega}_{2T} \;\big|\; \mathrm{d}(x,\Gamma) \le T - |t - T|\big\} \,.$$

Therewith, the choice of δ implies that $\widetilde{\boldsymbol{E}} = \widetilde{\boldsymbol{H}} = 0$ in $\widetilde{\Omega}_{b-\delta,T}$, proving (4.8).

Now we show that

$$\widetilde{\boldsymbol{E}}_{(\boldsymbol{j}_1,0)} \;=\; \widetilde{\boldsymbol{H}}_{(\boldsymbol{j}_1,0)} \;=\; 0 \qquad \text{in } \Omega \setminus \overline{B} \times (b - \delta, T) \,. \tag{4.9}$$

To this end we define

$$(\boldsymbol{E}_{\mathrm{com}}, \boldsymbol{H}_{\mathrm{com}}) \;:=\; \begin{cases} (\widetilde{\boldsymbol{E}}_{\boldsymbol{K}}, \widetilde{\boldsymbol{H}}_{\boldsymbol{K}}) & \text{in } B \times (b-\delta, T) \,,\\ (\widetilde{\boldsymbol{E}}_{(\boldsymbol{j}_1,0)}, \widetilde{\boldsymbol{H}}_{(\boldsymbol{j}_1,0)}) & \text{in } D \times (b-\delta, T) \,,\\ (\widetilde{\boldsymbol{E}}_{(\boldsymbol{j}_1,0)}, \widetilde{\boldsymbol{H}}_{(\boldsymbol{j}_1,0)}) \;=\; (\widetilde{\boldsymbol{E}}_{\boldsymbol{K}}, \widetilde{\boldsymbol{H}}_{\boldsymbol{K}}) & \text{in } \widetilde{\Omega}_{b-\delta,T} \,. \end{cases}$$

We argue as in the proof of Theorem 3.2 and appeal to the well-posedness of the backward homogeneous problem for Maxwell's equations to prove $\boldsymbol{E}_{\mathrm{com}} = \boldsymbol{H}_{\mathrm{com}} = 0$ in $\Omega_{b-\delta,T}$, which implies (4.9).

Step II: We use Lemma 4.4 below with $\sigma = b-\delta$ and $\tau = \delta$ to choose $\widetilde{\boldsymbol{j}}_1 \in \boldsymbol{L}^2(B_{b-\delta,b})$ such that $\widetilde{\boldsymbol{E}}_{(\widetilde{\boldsymbol{j}}_1,0)} \not\equiv 0$ in $(\Omega\setminus\overline{B})\times(b-\delta,b)$. Here $(\widetilde{\boldsymbol{E}}_{(\boldsymbol{j}_1,0)}, \widetilde{\boldsymbol{H}}_{(\boldsymbol{j}_1,0)})$ denotes the solution to (4.4) with source term $(\varepsilon(x)\mathcal{S}_T[\boldsymbol{j}_1],0)$. By construction,

$$\boldsymbol{Z}(x,t) \;:=\; -\mathcal{S}_0\big[\pi_\tau[\widetilde{\boldsymbol{H}}_{\boldsymbol{j}_1}]|_{\Gamma_T}\big] \in \operatorname{Ran}\mathbb{E}^*_{B_{a,b}}$$

However, $\boldsymbol{Z} \notin \operatorname{Ran}\mathbb{L}^*_{D_T}$ in view of Step I. Hence, we have established (4.7) and the proof is complete. □

In Step II of the proof of Theorem 4.3 we have used the following auxiliary result.

Lemma 4.4. *Consider an open subset $B \subset \Omega$ such that $\Omega\setminus\overline{B} \neq \emptyset$ and let $0 < \sigma < \sigma+\tau \le T$. Then there exists $\boldsymbol{j}_1 \in \boldsymbol{L}^2(B_{\sigma,\sigma+\tau})$ such that $\widetilde{\boldsymbol{E}}_{(\boldsymbol{j}_1,0)}|_{(\Omega\setminus\overline{B})\times(\sigma,\sigma+\tau)} \not\equiv 0$, where $(\widetilde{\boldsymbol{E}}_{(\boldsymbol{j}_1,0)}, \widetilde{\boldsymbol{H}}_{(\boldsymbol{j}_1,0)}) \in C([0,T];\boldsymbol{H}_0(\mathbf{curl};\Omega)\times\boldsymbol{H}(\mathbf{curl};\Omega)) \cap C^1([0,T];\boldsymbol{L}^2(\Omega)^2)$ denotes the unique weak solution of the IBVP (4.4) with source term $(\varepsilon(x)\mathcal{S}_T[\boldsymbol{j}_1],0)$. Moreover, such a $\boldsymbol{j}_1 \in \boldsymbol{L}^2(B_{\sigma,\sigma+\tau})$ can be constructed.*

Proof. Denoting $\mathcal{M}^{(\tau)} := \{x \in \Omega\setminus\overline{B} \mid \mathrm{d}(x,B) < \tau\}$, we introduce the operator

$$\mathbb{T}_{\sigma,\tau} : \boldsymbol{L}^2(B_{\sigma,\sigma+\tau}) \to \boldsymbol{L}^2(\mathcal{M}^{(\tau)}), \qquad \boldsymbol{j}_1 \mapsto \mathbb{P}_\tau\big[\widetilde{\boldsymbol{E}}_{\boldsymbol{j}_1}|_{t=\sigma}\big], \tag{4.10}$$

where $(\widetilde{\boldsymbol{E}}_{\boldsymbol{j}_1}, \widetilde{\boldsymbol{H}}_{\boldsymbol{j}_1})$ solves (4.4) for some $\boldsymbol{j}_1 \in \boldsymbol{L}^2(B_{\sigma,\sigma+\tau})$ and $\boldsymbol{j}_2 = 0$. Moreover, $\mathbb{P}_\tau$ denotes orthogonal projection from $\boldsymbol{L}^2(\mathcal{M}^{(\tau)})$ to its closed subspace $\boldsymbol{L}^2(\mathrm{div}_\varepsilon 0;\mathcal{M}^{(\tau)})$, defined by

$$\boldsymbol{L}^2(\mathrm{div}_\varepsilon 0;\mathcal{M}^{(\tau)}) \;:=\; \big\{\boldsymbol{g} \in \boldsymbol{L}^2(\mathcal{M}^{(\tau)}) \;\big|\; \mathrm{div}(\varepsilon\boldsymbol{g}) = 0 \quad \text{in } \Omega\big\}, \tag{4.11}$$

with respect to the inner product $\langle\cdot,\cdot\rangle_\varepsilon$ from (4.2).

Since $\boldsymbol{j}_1(\cdot,t) = 0$ for almost all $t \in (\sigma+\tau,T)$, we note that $\widetilde{\boldsymbol{E}}_{\boldsymbol{j}_1} = \widetilde{\boldsymbol{H}}_{\boldsymbol{j}_1} = 0$ in $\Omega_{\sigma+\tau,T}$. To prove the existence of the desired $\boldsymbol{j}_1 \in \boldsymbol{L}^2(B_{\sigma,\sigma+\tau})$, we prove a stronger result implying that $\operatorname{Ran}\mathbb{T}_{\sigma,\tau}$ is dense in $\boldsymbol{L}^2(\mathrm{div}_\varepsilon 0;\mathcal{M}^{(\tau)})$. To see this, we rely on a contrapositive argument. Suppose there exists $0 \neq \boldsymbol{\Phi} \in \boldsymbol{L}^2(\mathrm{div}_\varepsilon 0;\mathcal{M}^{(\tau)})$ such that

$$\int_{\mathcal{M}^{(\tau)}} \varepsilon(x)\mathbb{P}_\tau[\widetilde{\boldsymbol{E}}_{\boldsymbol{j}_1}|_{t=\sigma}](x)\cdot\boldsymbol{\Phi}(x)\,\mathrm{d}x \;=\; 0 \qquad \text{for all } \boldsymbol{j}_1 \in \boldsymbol{L}^2(B_{\sigma,\sigma+\tau}).$$

For this $\boldsymbol{\Phi}$, we now consider the unique weak solution $(\boldsymbol{E},\boldsymbol{H}) \in C([\sigma,\sigma+\tau];\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega))$ solving the IBVP

$$\begin{aligned}
\varepsilon(x)\partial_t\boldsymbol{E} - \mathbf{curl}\,\boldsymbol{H} \;&=\; 0 && \text{in } \Omega_{\sigma,\sigma+\tau}, && \text{(4.12a)}\\
\mu(x)\partial_t\boldsymbol{H} + \mathbf{curl}\,\boldsymbol{E} \;&=\; 0 && \text{in } \Omega_{\sigma,\sigma+\tau}, && \text{(4.12b)}\\
\boldsymbol{E}|_{t=\sigma} \;=\; \boldsymbol{\Phi}, \quad \boldsymbol{H}|_{t=\sigma} \;&=\; 0 && \text{in } \Omega, && \text{(4.12c)}\\
\boldsymbol{\nu}\times\boldsymbol{E}|_{(\partial\Omega)_{\sigma,\sigma+\tau}} \;&=\; 0 && \text{on } (\partial\Omega)_{\sigma,\sigma+\tau}, && \text{(4.12d)}
\end{aligned}$$

(see [2, Thm. 2 and Cor. 1]). Next, we consider a smooth approximation $\{\boldsymbol{\Phi}_k\}_{k\in\mathbb{N}} \in C_0^\infty(\Omega)$ of $\boldsymbol{\Phi}$ with $\boldsymbol{\Phi}_k \to \boldsymbol{\Phi}$ in $\boldsymbol{L}^2(\Omega)$ and denote by $(\boldsymbol{E}_k,\boldsymbol{H}_k) \in C([\sigma,\sigma+\tau];\boldsymbol{H}_0(\mathbf{curl};\Omega)\times\boldsymbol{H}(\mathbf{curl})) \cap C^1([\sigma,\sigma+\tau];\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega))$ the associated solution to (4.12) with $\boldsymbol{\Phi}_k$ instead of $\boldsymbol{\Phi}$ (see [22, Thm. 5.3]). Then, it follows from the continuous dependence of these solutions on the data (see [2, Thm. 2]) that

$$\boldsymbol{E} \;=\; \lim_{k\to\infty}\boldsymbol{E}_k, \qquad \boldsymbol{H} \;=\; \lim_{k\to\infty}\boldsymbol{H}_k \qquad \text{in } \boldsymbol{L}^2(\Omega_T).$$

Now, using orthogonality with respect to $\langle\cdot,\cdot\rangle_\varepsilon$ and the fact that $(\widetilde{\boldsymbol{E}}_{\boldsymbol{j}_1},\widetilde{\boldsymbol{H}}_{\boldsymbol{j}_1})$ vanishes at $t=\sigma+\tau$ and that $\boldsymbol{H}_k$ vanishes at $t=\sigma$ for all $k\in\mathbb{N}$, we find

$$\begin{aligned}
0 \;&=\; \int_{\mathcal{M}^{(\tau)}} \varepsilon(x)\mathbb{P}_\tau[\widetilde{\boldsymbol{E}}_{\boldsymbol{j}_1}|_{t=\sigma}](x)\cdot\boldsymbol{\Phi}(x)\;\mathrm{d}x \;=\; \int_{\mathcal{M}^{(\tau)}} \varepsilon(x)\widetilde{\boldsymbol{E}}_{\boldsymbol{j}_1}(x,\sigma)\cdot\boldsymbol{\Phi}(x)\;\mathrm{d}x\\
&=\; \lim_{k\to\infty}\int_{\mathcal{M}^{(\tau)}} \varepsilon(x)\widetilde{\boldsymbol{E}}_{\boldsymbol{j}_1}(x,\sigma)\cdot\boldsymbol{\Phi}_k(x)\;\mathrm{d}x\\
&=\; \lim_{k\to\infty}\int_\sigma^{\sigma+\tau}\int_\Omega \bigl(\varepsilon(x)\partial_t(\boldsymbol{E}_k\cdot\widetilde{\boldsymbol{E}}_{\boldsymbol{j}_1})(x,t)+\mu(x)\partial_t(\boldsymbol{H}_k\cdot\widetilde{\boldsymbol{H}}_{\boldsymbol{j}_1})(x,t)\bigr)\;\mathrm{d}x\;\mathrm{d}t\,.
\end{aligned}$$

Using (4.4) and (4.12), and integrating by parts with respect to x together with the homogeneous boundary conditions gives

$$\begin{aligned}
0 \;&=\; \lim_{k\to\infty}\int_\sigma^{\sigma+\tau}\int_\Omega \bigl(\mathbf{curl}\,\boldsymbol{H}_k\cdot\widetilde{\boldsymbol{E}}_{\boldsymbol{j}_1}+\boldsymbol{E}_k\cdot\mathbf{curl}\,\widetilde{\boldsymbol{H}}_{\boldsymbol{j}_1}-\boldsymbol{H}_k\cdot\mathbf{curl}\,\widetilde{\boldsymbol{E}}_{\boldsymbol{j}_1}-\widetilde{\boldsymbol{H}}_{\boldsymbol{j}_1}\cdot\mathbf{curl}\,\boldsymbol{E}_k\bigr)\;\mathrm{d}x\;\mathrm{d}t\\
&\quad+\lim_{k\to\infty}\int_\sigma^{\sigma+\tau}\int_\Omega \bigl(\varepsilon(x)\mathcal{S}_T[\boldsymbol{j}_1]\bigr)\cdot\boldsymbol{E}_k\;\mathrm{d}x\;\mathrm{d}t\\
&=\; \int_\sigma^{\sigma+\tau}\int_\Omega \bigl(\varepsilon(x)\mathcal{S}_T[\boldsymbol{j}_1]\bigr)\cdot\boldsymbol{E}\;\mathrm{d}x\;\mathrm{d}t\,,
\end{aligned}$$

where we omitted the arguments (x,t) for better readability. Recalling that $\boldsymbol{j}_1\in\boldsymbol{L}^2(B_{\sigma,\sigma+\tau})$ vanishes outside $(\sigma,\sigma+\tau)$, integration by parts with respect to t yields

$$\begin{aligned}
0 \;&=\; \int_\sigma^{\sigma+\tau}\int_\Omega \bigl(\varepsilon(x)\mathcal{S}_{\sigma+\tau}[\boldsymbol{j}_1]\bigr)\cdot\partial_t\bigl(\mathcal{S}_\sigma\boldsymbol{E}\bigr)\;\mathrm{d}x\;\mathrm{d}t\\
&=\; -\int_\sigma^{\sigma+\tau}\int_\Omega \boldsymbol{j}_1(x,t)\cdot\bigl(\varepsilon(x)\mathcal{S}_\sigma[\boldsymbol{E}](x,t)\bigr)\;\mathrm{d}x\;\mathrm{d}t\,.
\end{aligned}\tag{4.13}$$

Since (4.13) holds for all $\boldsymbol{j}_1\in\boldsymbol{L}^2(B_{\sigma,\sigma+\tau})$, we have shown that $\mathcal{S}_\sigma[\boldsymbol{E}]=0$ in $B_{\sigma,\sigma+\tau}$, which further implies that $\boldsymbol{E}=0$ in $B_{\sigma,\sigma+\tau}$. Thus also $\boldsymbol{H}=0$ in $B_{\sigma,\sigma+\tau}$ by (4.12).

Now we consider an extension of $(\boldsymbol{E},\boldsymbol{H})$ to $\Omega_{\sigma-\tau,\sigma+\tau}$ as follows,

$$(\boldsymbol{E}_{\mathrm{ex}}(x,t),\boldsymbol{H}_{\mathrm{ex}}(x,t)) \;=\; \begin{cases}(\boldsymbol{E}(x,t),\boldsymbol{H}(x,t))\,, & x\in\Omega\,,\; t\in[\sigma,\sigma+\tau)\,,\\ (\boldsymbol{E}(x,2\sigma-t),-\boldsymbol{H}(x,2\sigma-t))\,, & x\in\Omega\,,\; t\in(\sigma-\tau,\sigma)\,.\end{cases}$$

Note that this extension satisfies $(\boldsymbol{E}_{\mathrm{ex}},\boldsymbol{H}_{\mathrm{ex}})\in C([\sigma-\tau,\sigma+\tau];\boldsymbol{L}^2(\Omega)^2)\cap H^1([\sigma-\tau,\sigma+\tau];\boldsymbol{H}_0(\mathbf{curl};\Omega)^*\times\boldsymbol{H}(\mathbf{curl};\Omega)^*)$ and constitutes a weak solution of

$$\begin{aligned}
\varepsilon(x)\partial_t\boldsymbol{E}_{\mathrm{ex}}-\mathbf{curl}\,\boldsymbol{H}_{\mathrm{ex}} \;&=\; 0 &&\text{in } \Omega_{\sigma-\tau,\sigma+\tau}\,,\\
\mu(x)\partial_t\boldsymbol{H}_{\mathrm{ex}}+\mathbf{curl}\,\boldsymbol{E}_{\mathrm{ex}} \;&=\; 0 &&\text{in } \Omega_{\sigma-\tau,\sigma+\tau}\,,\\
\boldsymbol{E}_{\mathrm{ex}}|_{t=\sigma} \;=\; \boldsymbol{\Phi}\,,\quad \boldsymbol{H}_{\mathrm{ex}}|_{t=\sigma} \;&=\; 0 &&\text{in } \Omega\,,\\
\boldsymbol{\nu}\times\boldsymbol{E}_{\mathrm{ex}}|_{(\partial\Omega)_{\sigma,\sigma+\tau}} \;&=\; 0 &&\text{on } (\partial\Omega)_{\sigma,\sigma+\tau}\,.
\end{aligned}$$

We observe that the vanishing of $\boldsymbol{H}_{\mathrm{ex}}|_{t=\sigma}$ in Ω and the fact that $\boldsymbol{\Phi}\in\boldsymbol{L}^2(\mathrm{div}_\varepsilon 0;\mathcal{M}^{(\tau)})$ imply

$$\mathrm{div}(\varepsilon\boldsymbol{E}_{\mathrm{ex}}) \;=\; \mathrm{div}(\mu\boldsymbol{H}_{\mathrm{ex}}) \;=\; 0 \qquad \text{in } \Omega_{\sigma-\tau,\sigma+\tau}\,,$$

which can be seen similar to [2, Pro. 1]. Since

$$\boldsymbol{E}_{\mathrm{ex}} \;=\; \boldsymbol{H}_{\mathrm{ex}} \;=\; 0 \qquad \text{in } B_{\sigma-\tau,\sigma+\tau}\,,$$

invoking a unique continuation principle for Maxwell's equations [6, Thm. 4.5] (see also [7, Thm. 1.1]), we conclude that

$$\boldsymbol{E}_{\rm ex}(x,t) \;=\; \boldsymbol{H}_{\rm ex}(x,t) \;=\; 0 \qquad \text{in } \left\{(x,t)\in\Omega_{\sigma-\tau,\sigma+\tau} \;\middle|\; \mathrm{d}(x,\partial B)\le \tau-|t-\sigma|\right\}.$$

In particular, $\boldsymbol{\Phi}=\boldsymbol{E}_{\rm ex}|_{t=\sigma}=0$ in $\mathcal{M}^{(\tau)}$. This contradicts our choice of $\boldsymbol{\Phi}$, proving that $\operatorname{Ran}\mathbb{T}_{\sigma,\tau}$ is indeed dense in $\boldsymbol{L}^2(\operatorname{div}_\varepsilon 0;\mathcal{M}^{(\tau)})$. Therefore, the existence of the desired $\boldsymbol{j}_1\in\boldsymbol{L}^2(B^{(1)}_{a,b})$ for Lemma 4.4 is guaranteed.

For the construction of such a source $\boldsymbol{j}_1$, we can rely on a minimization argument based on a Tikhonov functional as done in the proof of Lemma 3.3. Without repeating the steps of the proof, we argue that

$$\boldsymbol{j}_1 \;:=\; \left(\mathbb{T}^*_{\sigma,\tau}\mathbb{T}_{\sigma,\tau}+\beta_0\mathbb{I}\right)^{-1}\mathbb{T}^*_{\sigma,\tau}[\boldsymbol{g}]$$

is an example for the desired source in Lemma 4.4 for $0\neq\boldsymbol{g}\in\boldsymbol{L}^2(\operatorname{div}_\varepsilon 0;\mathcal{M}^{(\tau)})$ and $\beta_0>0$ sufficiently small. This concludes the proof of Lemma 4.4. □

Next we consider the construction of localized waves as in Theorem 4.3.

Corollary 4.5. *A sequence of boundary sources $\{\boldsymbol{f}_k\}_{k\in\mathbb{N}}\subset\mathcal{H}^1_0([0,T];\widetilde{H}^{-1/2}(\mathrm{Div};\Gamma))$ as in Theorem 4.3 can be explicitly constructed as*

$$\boldsymbol{f}_k \;=\; \frac{1}{\|\sqrt{\mathbb{J}_k}[\xi]\|^{3/2}}\mathbb{J}_k[\boldsymbol{\xi}] \quad \textit{with} \quad \mathbb{J}_k \;:=\; \left(\mathbb{L}^*_{D_T}\mathbb{L}_{D_T}+k^{-1}\mathbb{I}\right)^{-1}, \qquad k\in\mathbb{N},$$

where, taking $\beta>0$ small and $0<\tau<\min\{b-a,b-\mathrm{d}(\Omega,\Gamma)\}$, we choose $\sigma=b-\tau$ and $0\neq\widetilde{\boldsymbol{g}}\in\boldsymbol{L}^2(\operatorname{div}_\varepsilon 0;\mathcal{M}^{(\tau)})$ to define

$$\boldsymbol{\xi} \;:=\; \mathbb{E}^*_{B_{a,b}}\left[\left(\mathbb{T}^*_{\sigma,\tau}\mathbb{T}_{\sigma,\tau}+\beta\mathbb{I}\right)^{-1}\mathbb{T}^*_{\sigma,\tau}[\widetilde{\boldsymbol{g}}]\right].$$

Here the operators $\mathbb{E}_{B_{a,b}}$, $\mathbb{L}_{D_T}$, and $\mathbb{T}_{\sigma,\tau}$ are defined in (4.1), and (4.10).

Proof. To construct $\{\boldsymbol{f}_k\}_{k\in\mathbb{N}}$, we appeal to Lemma A.2 with the choice $\mathcal{A}_1=\mathbb{E}^*_{B_{a,b}}$, $\mathcal{A}_2=\mathbb{L}^*_{D_T}$, and $\boldsymbol{\xi}=\mathbb{E}^*_{B_{a,b}}\left[\widetilde{\boldsymbol{j}}_1\right]$. For $\sigma=b-\tau$ with $\tau<\min\{b-a,\,\mathrm{d}(\Omega,\Gamma)\}$, we consider here

$$\widetilde{\boldsymbol{j}}_1 \;:=\; \left(\mathbb{T}^*_{\sigma,\tau}\mathbb{T}_{\sigma,\tau}+\beta\mathbb{I}\right)^{-1}\mathbb{T}^*_{\sigma,\tau}[\widetilde{\boldsymbol{g}}]$$

after fixing some $0\neq\widetilde{\boldsymbol{g}}\in\boldsymbol{L}^2(\operatorname{div}_\varepsilon 0;\mathcal{M}^{(\tau)})$ and $\beta>0$ sufficiently small. The operator $\mathbb{T}_{\sigma,\tau}$ is introduced in (4.10). From our choice of $\widetilde{\boldsymbol{j}}_1$ and the arguments of Step II of Theorem 4.3, it is clear that $\boldsymbol{\xi}\notin\operatorname{Ran}\mathbb{L}^*_{D_T}$. Similar to the discussion in Corollary 3.4, we note that $\sqrt{\mathbb{J}_k}$ makes sense since $\mathbb{J}_k$ is positive. Denoting $\boldsymbol{\eta}_k=\mathbb{J}_k[\boldsymbol{\xi}]$, we see $\langle\boldsymbol{\xi},\boldsymbol{\eta}_k\rangle=\langle\boldsymbol{\xi},\mathbb{J}_k[\boldsymbol{\xi}]\rangle=\|\sqrt{\mathbb{J}_k}[\boldsymbol{\xi}]\|^2$ for $k\in\mathbb{N}$. With this observation, an application of Lemma A.2 concludes the proof of Corollary 4.5. □

4.2 Localization in time

Similar to Section 3.2, we can also construct a sequence of boundary data for which the associated solutions to (2.2) concentrates on any given open set $B\subset\Omega$ at a sufficiently large time.

Theorem 4.6. *Suppose, in addition to our previous assumptions, that $\varepsilon,\mu\in C^2(\overline{\Omega})$. Consider an open subset $B\Subset\Omega$ such that $\Omega\setminus\overline{B}$ is connected, and let $[a,b]$ and $[c,d]$ be two subintervals of $[0,T]$ with $[a,b]\cap[c,d]=\emptyset$ such that*

(I) $\mathrm{d}(\Omega,\Gamma)<|b-d|$ *if $c<a$,*

(II) $\mathrm{d}(\Omega,\Gamma) < b$ *and* $\mathrm{d}(B,\partial B) < |b-c|/2$ *if* $c > a$.

Then, there exist sequences $\{\boldsymbol{f}_k\}_{k\in\mathbb{N}}, \{\widetilde{\boldsymbol{f}}_k\}_{k\in\mathbb{N}} \subset \mathcal{H}^1_0([0,T];\widetilde{H}^{-1/2}(\mathrm{Div};\Gamma))$ *such that*

$$\begin{aligned} &\|\boldsymbol{E}_{\boldsymbol{f}_k}\|_{\boldsymbol{L}^2(B_{a,b})} \to \infty \quad \text{and} \quad \|\boldsymbol{E}_{\boldsymbol{f}_k}\|_{L^2(B_{c,d})} + \|\boldsymbol{H}_{\boldsymbol{f}_k}\|_{L^2(B_{c,d})} \to 0 \quad \text{as } k\to\infty\,,\\ &\|\boldsymbol{H}_{\widetilde{\boldsymbol{f}}_k}\|_{\boldsymbol{L}^2(B_{a,b})} \to \infty \quad \text{and} \quad \|\boldsymbol{E}_{\widetilde{\boldsymbol{f}}_k}\|_{\boldsymbol{L}^2(B_{c,d})} + \|\boldsymbol{H}_{\widetilde{\boldsymbol{f}}_k}\|_{\boldsymbol{L}^2(B_{c,d})} \to 0 \quad \text{as } k\to\infty\,. \end{aligned}$$

Here, $(\boldsymbol{E}_{\boldsymbol{f}_k}, \boldsymbol{H}_{\boldsymbol{f}_k})$ *and* $(\boldsymbol{E}_{\widetilde{\boldsymbol{f}}_k}, \boldsymbol{H}_{\widetilde{\boldsymbol{f}}_k})$ *denote the solutions to* (2.2) *for* $\boldsymbol{f} = \boldsymbol{f}_k$ *and* $\boldsymbol{f} = \widetilde{\boldsymbol{f}}_k$*, respectively.*

Proof. We divide the details of proof of Theorem 4.6 in two cases. Our proof relies on an argument similar to that of Theorem 3.8. Also, we only discuss the existence and construction of $\boldsymbol{f}_k$ and choose to avoid the corresponding discussion of $\widetilde{\boldsymbol{f}}_k$. The latter only requires replacing $\mathbb{E}^*_{B_{a,b}}$ by $\mathbb{H}^*_{B_{a,b}}$ and adjusting Lemma 4.7 appropriately in the following argument.

Case I: (c < a) The intervals $[a,b]$ and $[c,d]$ being disjoint, we see that $d < a$. The discussion beyond $t = b$ being immaterial to us, it suffices to construct boundary sources $\boldsymbol{f}_k \in \mathcal{H}^1_0([0,b];\widetilde{H}^{-1/2}(\mathrm{Div};\Gamma))$. From Lemma 4.7 below with $\tau = b-d$, we obtain $\widetilde{\boldsymbol{f}} \in \mathcal{H}^1_0([0,b-d];\widetilde{H}^{-1/2}(\mathrm{Div};\Gamma))$, such that $\boldsymbol{E}_{\widetilde{\boldsymbol{f}}}|_{t=b-d} \not\equiv 0$ in B. Here, we have utilized our assumption that $\mathrm{d}(\Omega,\Gamma) < |b-d|$. Defining $\boldsymbol{f} := \mathcal{T}_d[\widetilde{\boldsymbol{f}}]$ by time-translation as in (3.1), and extending this function by zero to $[0,T]$, gives $\boldsymbol{f} \in \mathcal{H}^1_0([0,T];\widetilde{H}^{-1/2}(\mathrm{Div};\Gamma))$ such that the solution $(\boldsymbol{E}_{\boldsymbol{f}}, \boldsymbol{H}_{\boldsymbol{f}})$ to the IBVP (2.2) satisfies $\boldsymbol{E}_{\boldsymbol{f}}(x,t) = \boldsymbol{H}_{\boldsymbol{f}}(x,t) = 0$ for $(x,t) \in \Omega_d$ and $\boldsymbol{E}_{\boldsymbol{f}}|_{t=b} \not\equiv 0$ in B. Since $\boldsymbol{E}_{\boldsymbol{f}} \in C([0,b];\boldsymbol{L}^2(\Omega))$, we have $\|\boldsymbol{E}_{\boldsymbol{f}}\|_{\boldsymbol{L}^2(B_{a,b})} \neq 0$. This implies that the sequence of boundary data defined by $\boldsymbol{f}_k = k\boldsymbol{f}$ for $k \in \mathbb{N}$ validates Theorem 4.6.

Case II: (a < c) From our assumption that $[a,b]\cap[c,d] = \emptyset$, we also have $b < c$. In view of Lemma A.2, it suffices to show

$$\mathrm{Ran}\,\mathbb{E}^*_{B_{a,b}} \cap \mathrm{Ran}\,\mathbb{L}^*_{B_{c,d}} = \{0\} \tag{4.14}$$

in order to prove Theorem 4.6. As before, we use a contrapositive argument to establish (4.14).

Let us assume $\boldsymbol{h} \in \mathrm{Ran}\,\mathbb{E}^*_{B_{a,b}} \cap \mathrm{Ran}\,\mathbb{L}^*_{B_{c,d}}$ implying

$$\boldsymbol{h} = -\mathcal{S}_0\big[\pi_\tau[\widetilde{\boldsymbol{H}}_{(\boldsymbol{j}_1,0)}]\big|_{\Gamma_T}\big] = -\mathcal{S}_0\big[\pi_\tau[\widetilde{\boldsymbol{H}}_{\boldsymbol{K}}]\big|_{\Gamma_T}\big]\,,$$

where $(\widetilde{\boldsymbol{E}}_{(\boldsymbol{j}_1,0)}, \widetilde{\boldsymbol{H}}_{(\boldsymbol{j}_1,0)})$ and $(\widetilde{\boldsymbol{E}}_{\boldsymbol{K}}, \widetilde{\boldsymbol{H}}_{\boldsymbol{K}})$ solve the IBVP (4.4) with source terms $(\varepsilon(x)\mathcal{S}_T[\boldsymbol{j}_1],0)$ and $(\varepsilon(x)\mathcal{S}_T[\boldsymbol{k}_1], \mu(x)\mathcal{S}_T[\boldsymbol{k}_2])$ for some $\boldsymbol{j}_1 \in \boldsymbol{L}^2(B_{a,b})$ and $\boldsymbol{K} = (\boldsymbol{k}_1,\boldsymbol{k}_2) \in \boldsymbol{L}^2(B_{c,d})^2$, respectively. As we have seen in Remark 4.2, we can assume without loss of generality that $\mathrm{div}(\varepsilon\boldsymbol{k}_1) = \mathrm{div}(\mu\boldsymbol{k}_2) = 0$. Since $\boldsymbol{j}_1 = 0$ and $\boldsymbol{K} = (0,0)$ in $B_{d,T}$, we obtain $(\widetilde{\boldsymbol{E}}_{(\boldsymbol{j}_1,0)}, \widetilde{\boldsymbol{H}}_{(\boldsymbol{j}_1,0)}) = (\widetilde{\boldsymbol{E}}_{\boldsymbol{K}}, \widetilde{\boldsymbol{H}}_{\boldsymbol{K}}) = (0,0)$ in $\Omega_{d,T}$.

We denote $\widetilde{\Omega} := \Omega \setminus \overline{B}$ and define $\boldsymbol{E} := \widetilde{\boldsymbol{E}}_{(\boldsymbol{j}_1,0)} - \widetilde{\boldsymbol{E}}_{\boldsymbol{K}}$ and $\boldsymbol{H} := \widetilde{\boldsymbol{H}}_{(\boldsymbol{j}_1,0)} - \widetilde{\boldsymbol{H}}_{\boldsymbol{K}}$ in Ω_d. Extending $(\boldsymbol{E},\boldsymbol{H})$ from Ω_d to Ω_{2d} by zero and denoting this extension again by $(\boldsymbol{E},\boldsymbol{H})$, we see that $(\boldsymbol{E},\boldsymbol{H}) \in C([0,2d];\boldsymbol{H}_0(\mathbf{curl};\Omega)\times\boldsymbol{H}(\mathbf{curl};\Omega)) \cap C^1([0,2d];\boldsymbol{L}^2(\Omega)^2)$ due to Lemma 4.1. Note that $(\boldsymbol{E},\boldsymbol{H})$ satisfies

$$\begin{aligned} \varepsilon(x)\partial_t\boldsymbol{E} - \mathbf{curl}\,\boldsymbol{H} &= 0 && \text{in } \widetilde{\Omega}_{2d}\,,\\ \mu(x)\partial_t\boldsymbol{H} + \mathbf{curl}\,\boldsymbol{E} &= 0 && \text{in } \widetilde{\Omega}_{2d}\,,\\ \boldsymbol{E}|_{t=d} = \boldsymbol{H}|_{t=d} &= 0 && \text{in } \widetilde{\Omega}\,,\\ \nu\times\boldsymbol{E}|_{\Gamma_{2d}} = \nu\times\boldsymbol{H}|_{\Gamma_{2d}} &= 0 && \text{on } \Gamma_{2d}\,, \end{aligned}$$

and thus also

$$\mathrm{div}(\varepsilon(x)\boldsymbol{E}) = \mathrm{div}(\mu(x)\boldsymbol{H}) = 0 \qquad \text{in } \widetilde{\Omega}_{2d}\,.$$

In view of unique continuation for Maxwell's equations [6, Thm. 4.5] (see also [7, Thm. 1.1]), we can conclude that

$$\boldsymbol{E} = \boldsymbol{H} = 0 \qquad \text{in } \left\{(x,t) \in \widetilde{\Omega}_{2d} \;\middle|\; \mathrm{d}(x,\Gamma) \le d - |t-d|\right\},$$

which in consideration of our assumption $\mathrm{d}(\Omega,\Gamma) < b$ implies that

$$\boldsymbol{E}(x,t) = \boldsymbol{H}(x,t) = 0 \qquad \text{for } (x,t) \in \widetilde{\Omega}_{b,d}\,.$$

Since $\boldsymbol{j}_1(\cdot,t) = 0$ for almost every $t \in (b,T)$, we also have $(\widetilde{\boldsymbol{E}}_{(\boldsymbol{j}_1,0)}, \widetilde{\boldsymbol{H}}_{(\boldsymbol{j}_1,0)}) = (0,0)$ in $\Omega_{b,d}$, which implies $\widetilde{\boldsymbol{E}}_{\boldsymbol{K}} = \widetilde{\boldsymbol{H}}_{\boldsymbol{K}} = 0$ in $\widetilde{\Omega}_{b,d}$. To see that $\boldsymbol{h} = 0$ on Γ_T, it is therefore enough to prove $\pi_\tau[\widetilde{\boldsymbol{H}}_{\boldsymbol{K}}] = 0$ on Γ_b. This is what we show next.

Let us start with the observation that

$$\begin{aligned}
\varepsilon(x)\partial_t\widetilde{\boldsymbol{E}}_{\boldsymbol{K}} - \mathbf{curl}\,\widetilde{\boldsymbol{H}}_{\boldsymbol{K}} &= -\int_c^T \varepsilon(x)\boldsymbol{k}_1(x,s)\,\mathrm{d}s && \text{in } \Omega_{b,c}\,,\\
\mu(x)\partial_t\widetilde{\boldsymbol{H}}_{\boldsymbol{K}} + \mathbf{curl}\,\widetilde{\boldsymbol{E}}_{\boldsymbol{K}} &= -\int_c^T \mu(x)\boldsymbol{k}_2(x,s)\,\mathrm{d}s && \text{in } \Omega_{b,c}\,.
\end{aligned}$$

In particular the source terms do not depend on time. Accordingly, we define

$$\widetilde{\boldsymbol{E}}'_{\boldsymbol{K}}(x,t) := \partial_t\widetilde{\boldsymbol{E}}_{\boldsymbol{K}}(x,t) \qquad \text{and} \qquad \widetilde{\boldsymbol{H}}'_{\boldsymbol{K}}(x,t) := \partial_t\widetilde{\boldsymbol{H}}_{\boldsymbol{K}}(x,t)\,,$$

and we notice that $\operatorname{div}(\varepsilon(x)\widetilde{\boldsymbol{E}}'_{\boldsymbol{K}}) = \operatorname{div}(\mu(x)\widetilde{\boldsymbol{H}}'_{\boldsymbol{K}}) = 0$ in $\Omega_{b,c}$ and $\widetilde{\boldsymbol{E}}'_{\boldsymbol{K}} = \widetilde{\boldsymbol{H}}'_{\boldsymbol{K}} = 0$ in $\widetilde{\Omega}_{b,c}$. Moreover, $(\widetilde{\boldsymbol{E}}'_{\boldsymbol{K}}, \widetilde{\boldsymbol{H}}'_{\boldsymbol{K}}) \in C([0,T];\boldsymbol{L}^2(\Omega)^2)$. We would like to invoke the unique continuation result from [6, 7], which, however, requires more regularity. For this, one can mollify $\widetilde{\boldsymbol{E}}'_{\boldsymbol{K}}$ and $\widetilde{\boldsymbol{H}}'_{\boldsymbol{K}}$ in time. Although this is standard, we briefly discuss this here for completeness. Taking $\gamma \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\gamma \subset [-1,1]$ and $\int_{\mathbb{R}} \gamma\,\mathrm{d}t = 1$, let us consider

$$\widetilde{\boldsymbol{E}}'_{\boldsymbol{K},n} = \gamma_n *_t \boldsymbol{E}'_{\boldsymbol{K}}\,, \quad \widetilde{\boldsymbol{H}}'_{\boldsymbol{K},n} = \gamma_n *_t \boldsymbol{H}'_{\boldsymbol{K}} \qquad \text{where } \gamma_n(t) = n\gamma(nt) \text{ for } t \in \mathbb{R},\ n \in \mathbb{N}\,,$$

and $*_t$ denotes a convolution in time. Taking $n \in \mathbb{N}$ large enough, we may assume that $\mathrm{d}(B,\partial B) < |c_n - b_n|/2$ where $b_n := b + \frac{1}{n}$ and $c_n := c - \frac{1}{n}$. In fact, we note that $\widetilde{\boldsymbol{E}}'_{\boldsymbol{K},n} = \widetilde{\boldsymbol{H}}'_{\boldsymbol{K},n} = 0$ in $\widetilde{\Omega}_{b,d}$, and $(\widetilde{\boldsymbol{E}}'_{\boldsymbol{K},n}, \widetilde{\boldsymbol{H}}'_{\boldsymbol{K},n}) \in \boldsymbol{H}^2(\Omega_{b_n,c_n})$ satisfies the homogeneous Maxwell equations

$$\begin{aligned}
\varepsilon(x)\partial_t\widetilde{\boldsymbol{E}}'_{\boldsymbol{K},n} - \mathbf{curl}\ \widetilde{\boldsymbol{H}}'_{\boldsymbol{K},n} &= 0 && \text{in } \Omega_{b,c}\,,\\
\mu(x)\partial_t\widetilde{\boldsymbol{H}}'_{\boldsymbol{K},n} + \mathbf{curl}\ \widetilde{\boldsymbol{E}}'_{\boldsymbol{K},n} &= 0 && \text{in } \Omega_{b,c}\,.
\end{aligned}$$

The $\boldsymbol{H}^2$-regularity of $\widetilde{\boldsymbol{E}}'_{\boldsymbol{K},n}$ (and also $\widetilde{\boldsymbol{H}}'_{\boldsymbol{K},n}$) is achieved by employing elliptic regularity results since $\mathbf{curl}\,\mathbf{curl}\,\widetilde{\boldsymbol{E}}'_{\boldsymbol{K},n}(\cdot,t) \in \boldsymbol{L}^2(\Omega)$ and $\operatorname{div}\widetilde{\boldsymbol{E}}'_{\boldsymbol{K},n}(\cdot,t) \in \boldsymbol{L}^2(\Omega)$ for all $t \in [b_n,c_n]$. For the latter we use the C^2-regularity of ε along with $\operatorname{div}(\varepsilon(x)\widetilde{\boldsymbol{E}}'_{\boldsymbol{K},n}) = 0$ in Ω_{b_n,c_n}. Recalling that $\widetilde{\boldsymbol{E}}'_{\boldsymbol{K},n} = \widetilde{\boldsymbol{H}}'_{\boldsymbol{K},n} = 0$ in $\widetilde{\Omega}_{b,d}$, we appeal to the unique continuation principle [6, Thm. 4.5] applied to $(\boldsymbol{E}'_{\boldsymbol{K},n}, \boldsymbol{H}'_{\boldsymbol{K},n})$ (see also [7, Thm.1.1]) to see that

$$\boldsymbol{E}'_{\boldsymbol{K},n} = \boldsymbol{H}'_{\boldsymbol{K},n} = 0 \qquad \text{in } \left\{(x,t) \in B_{b,c} \;\middle|\; \mathrm{d}(x,\partial B) < \frac{c-b}{2} - \frac{1}{n} + \left|t - \frac{c+b}{2}\right|\right\}.$$

Since $(\boldsymbol{E}'_{\boldsymbol{K},n}, \boldsymbol{H}'_{\boldsymbol{K},n}) \to (\boldsymbol{E}'_{\boldsymbol{K}}, \boldsymbol{H}'_{\boldsymbol{K}})$ in $\boldsymbol{L}^2(\Omega_{b,c})^2$ as $n \to \infty$, we have $\widetilde{\boldsymbol{E}}'_{\boldsymbol{K}} = \widetilde{\boldsymbol{H}}'_{\boldsymbol{K}} = 0$ a.e. in $B \times \{\frac{b+c}{2}\}$. Combining this with $\widetilde{\boldsymbol{E}}'_{\boldsymbol{K}} = \widetilde{\boldsymbol{H}}'_{\boldsymbol{K}} = 0$ in $\widetilde{\Omega}_{b,c}$, we therefore have that $\widetilde{\boldsymbol{E}}'_{\boldsymbol{K}} = \widetilde{\boldsymbol{H}}'_{\boldsymbol{K}} = 0$

in $\Omega_{\frac{b+c}{2}}$ from the well-posedness of IBVP satisfied by $(\widetilde{\boldsymbol{E}}'_{\boldsymbol{K}}, \widetilde{\boldsymbol{H}}'_{\boldsymbol{K}})$ in $\Omega_{\frac{b+c}{2}}$. Since $\widetilde{\boldsymbol{E}}_{\boldsymbol{K}} = \widetilde{\boldsymbol{H}}_{\boldsymbol{K}} = 0$ on $\widetilde{\Omega} \times \{\frac{b+c}{2}\}$ as well, we obtain by integrating with respect to time that $\widetilde{\boldsymbol{E}}_{\boldsymbol{K}} = \widetilde{\boldsymbol{H}}_{\boldsymbol{K}} = 0$ in $\Omega_{\frac{b+c}{2}}$. This further gives $\pi_\tau[\widetilde{\boldsymbol{H}}_{\boldsymbol{K}}] = 0$ on Γ_b, and the proof of (4.14) is complete.

For the construction of $\boldsymbol{f}_k$, we appeal to Lemma A.2 with a choice of $0 \neq \boldsymbol{\xi} \in \operatorname{Ran} \mathbb{E}^*_{B_{a,b}}$. In view of the arguments presented above, we note that $\boldsymbol{\xi} \notin \operatorname{Ran} \mathbb{L}^*_{B_{c,d}}$. In order to choose such a $\boldsymbol{\xi}$, we now refer to Lemma 4.4 after fixing some $0 \neq \widetilde{\boldsymbol{g}} \in \boldsymbol{L}^2(\operatorname{div}_\varepsilon 0; \mathcal{M}^{(\tau)})$. Taking $\sigma = b - \tau$ and $\tau < \min\{b - a, \, \mathrm{d}(B, \partial\Omega)\}$ in Lemma 4.4, we consider $\widetilde{\boldsymbol{j}}_1 \in L^2(B_{a,b})$ defined by

$$\widetilde{\boldsymbol{j}}_1 \,:=\, \big(\mathbb{T}^*_{\sigma,\tau}\mathbb{T}_{\sigma,\tau} + \beta_0 \mathbb{I}\big)^{-1}\mathbb{T}^*_{\sigma,\tau}[\widetilde{\boldsymbol{g}}] \,.$$

In consideration of such $\widetilde{\boldsymbol{j}}_1$, the preceding arguments also yield that the choice $\boldsymbol{\xi} := \mathbb{L}^*_{B_{a,b}}[\widetilde{\boldsymbol{j}}_1]$ satisfies $\boldsymbol{\xi} \neq 0$. Now we consider

$$\boldsymbol{f}_k \,:=\, \frac{1}{\|\sqrt{\mathbb{J}_k}[\boldsymbol{\xi}]\|^{3/2}} \mathbb{J}_k[\boldsymbol{\xi}] \quad \text{with} \quad \mathbb{J}_k \,:=\, \big(\mathbb{L}^*_{B_{c,d}}\mathbb{L}_{B_{c,d}} + k^{-1}\mathbb{I}\big)^{-1} \,, \qquad k \in \mathbb{N} \,,$$

Once again, we denote $\boldsymbol{\eta}_k := \mathbb{J}_k[\boldsymbol{\xi}]$ and then use the positivity of $\mathbb{J}_k$ to make sense of $\sqrt{\mathbb{J}_k}$ while applying Lemma A.2. Here, we have also used the relation $\langle \boldsymbol{\xi}, \boldsymbol{\eta}_k \rangle = \langle \boldsymbol{\xi}, \mathbb{J}_k[\boldsymbol{\xi}] \rangle = \|\sqrt{\mathbb{J}_k}[\boldsymbol{\xi}]\|^2$ for $k \in \mathbb{N}$. Therefore, the construction of $\boldsymbol{f}_k$ as required in Case II is complete. □

In Case I of the proof of Theorem 4.6, we have used the following auxiliary result.

Lemma 4.7. *For $\tau > \mathrm{d}(\Omega, \Gamma)$ and $B \subseteq \Omega$ open, we can construct $\boldsymbol{f} \in \mathcal{H}^1_0([0,\tau]; \widetilde{H}^{-1/2}(\mathrm{Div}; \Gamma))$ such that $\boldsymbol{E}_{\boldsymbol{f}}|_{t=\tau} \not\equiv 0$ in B, where $(\boldsymbol{E}_{\boldsymbol{f}}, \boldsymbol{H}_{\boldsymbol{f}})$ denotes the solution to (2.2).*

We omit details of the proof of Lemma 4.7, as the arguments are similar to that of Lemma 3.9. Rather, we briefly discuss the construction aspect of $\boldsymbol{f} \in \mathcal{H}^1_0([0,\tau]; \widetilde{H}^{-1/2}(\mathrm{Div}; \Gamma))$ as in Lemma 4.7. Denoting $\boldsymbol{L}^2(\operatorname{div}_\varepsilon 0; \Omega)$ as defined in (4.11) with $\mathcal{M}^{(\tau)}$ replaced by Ω, we employ unique continuation of Maxwell's equations to see that the mapping

$$\mathbb{P}_\tau : \mathcal{H}^1_0([0,\tau]; \widetilde{H}^{-1/2}(\mathrm{Div}; \Gamma)) \to \boldsymbol{L}^2(\operatorname{div}_\varepsilon 0; \Omega) \,, \quad \boldsymbol{f} \mapsto \boldsymbol{E}_{\boldsymbol{f}}|_{t=\tau}$$

has dense range. Here $(\boldsymbol{E}_{\boldsymbol{f}}, \boldsymbol{H}_{\boldsymbol{f}})$ denotes the solution to (2.2) for $\tau = T$. In order to construct $\boldsymbol{f}$ such that $\boldsymbol{E}_f \neq 0$ in $B \subseteq \Omega$, we first fix $\boldsymbol{\Phi} \in \boldsymbol{L}^2(\operatorname{div}_\varepsilon 0; \Omega)$ such that $\boldsymbol{\Phi} \not\equiv \boldsymbol{0}$ in B. Note that this can be always done by taking $\boldsymbol{\Phi} = \varepsilon^{-1} \,\mathbf{curl}\, \boldsymbol{\Psi}$ where $\boldsymbol{\Psi} \in C^\infty_c(B)$ with $\mathbf{curl}\, \boldsymbol{\Psi} \not\equiv 0$ in B. With such choice of $\boldsymbol{\Phi}$, we rely on Tikhonov regularization as discussed in Lemma 3.3 to conclude that

$$\lim_{\beta \to 0+} \mathbb{P}_\tau[\boldsymbol{f}_\beta] \,=\, \boldsymbol{\Phi} \,, \qquad \text{where } \boldsymbol{f}_\beta \,:=\, \big(\mathbb{P}^*_\tau \mathbb{P}_\tau + \beta \mathbb{I}\big)^{-1} \mathbb{P}^*_\tau[\boldsymbol{\Phi}] \,.$$

Therefore, $\boldsymbol{f}_\beta$ provides an example of $\boldsymbol{f}$ as desired in Lemma 4.7 when $\beta > 0$ is sufficiently small.

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Appendix. An abstract functional analytic result.

Our construction of localized wave functions crucially relies on a functional analytic result which characterizes conditions for the range inclusion of two operators by means of comparing the pointwise norms of their adjoint operators. We recall this for the readers convenience.

Lemma A.1 (Prop. 12.1.2. of [34]). *For $i = 1, 2$, consider the bounded linear map $\mathcal{A}_i : Y_i \to X$ where X, Y_1 and Y_2 denote three Hilbert spaces. Then we have* $\operatorname{Ran}\mathcal{A}_1 \subseteq \operatorname{Ran}\mathcal{A}_2$ *if and only if there exists a $C > 0$ such that*

$$\|\mathcal{A}_1^*[\xi]\|_{Y_1} \le C\|\mathcal{A}_2^*[\xi]\|_{Y_2} \qquad \textit{for all } \xi \in X\,,$$

where $\mathcal{A}_i^*$ denotes the adjoint to $\mathcal{A}_i$ for $i = 1, 2$.

We further note that Lemma A.1 is valid even for reflexive Banach spaces. We refer the reader to consult [8, Lmm. 2.5] for a proof.

Next we provide a result regarding the construction of localized wave functions in the abstract framework of Lemma A.1. Such a construction has been already derived in [8, Lmm. 2.8] under the additional assumption that the adjoint operators in Lemma A.2, i.e., $\mathcal{A}_i^*$, $i = 1, 2$, are injective.

Lemma A.2. *Let $\mathcal{A}_i$, $i = 1, 2$, be the operators as defined in Lemma A.1. Suppose $\xi \in \operatorname{Ran}\mathcal{A}_1$ but $\xi \notin \operatorname{Ran}\mathcal{A}_2$. Then, we have $\lim_{\alpha\to 0+}\|\mathcal{A}_1^*[\xi_\alpha]\| = \infty$ and $\lim_{\alpha\to 0+}\mathcal{A}_2^*[\xi_\alpha] = 0$, where*

$$\xi_\alpha := \frac{\eta_\alpha}{\langle \xi, \eta_\alpha\rangle^{3/4}}\,, \quad \eta_\alpha := \left(\mathcal{A}_2\mathcal{A}_2^* + \alpha\mathbb{I}\right)^{-1}[\xi]\,, \quad \alpha > 0\,.$$

Proof. Note that, the definition of ξ_α makes sense since $\langle \xi, \eta_\alpha\rangle > 0$, which follows from the relation

$$\langle \xi, \eta_\alpha\rangle = \|\mathcal{A}_2^*[\eta_\alpha]\|^2 + \alpha\|\eta_\alpha\|^2\,, \qquad \alpha > 0 \tag{A.1}$$

and the fact that $\xi \neq 0$. This implies $\eta_\alpha \neq 0$ making the left hand side of (A.1) positive. In order to prove Lemma A.2, we start with the observation that

$$\lim_{\alpha\to 0+}\langle \xi, \eta_\alpha\rangle = \infty\,. \tag{A.2}$$

To see this, we assume the contrary, i.e., $\{\langle \xi, \eta_{\alpha_k}\rangle\}_{k\in\mathbb{N}}$ is bounded for some sequence $\alpha_k \to 0+$. Making use of (A.1), we therefore see that the sequences $\{\mathcal{A}_2^*[\eta_{\alpha_k}]\}_{k\in\mathbb{N}}$ and $\{\sqrt{\alpha_k}\,\eta_{\alpha_k}\}_{k\in\mathbb{N}}$ are bounded in Y_2 and X respectively. An application of the Banach-Alaoglu theorem implies that both these sequences admit weakly convergent subsequences which we may consider to be same up to a subsequence. With an abuse of notation, we still denote this subsequence by $\{\alpha_k\}_{k\in\mathbb{N}}$. Summarizing, we have

$$\sqrt{\alpha_k}\,\eta_{\alpha_k} \rightharpoonup \xi_0\,, \qquad \mathcal{A}_2^*[\eta_{\alpha_k}] \rightharpoonup \psi_2\,, \tag{A.3}$$

for some $\xi_0 \in X$ and $\psi_2 \in Y_2$. For any $\widetilde{\xi} \in X$, we use (A.3) to compute

$$\begin{aligned}\langle \widetilde{\xi}, \mathcal{A}_2[\psi_2]\rangle &= \lim_{k\to\infty}\langle \widetilde{\xi}, \mathcal{A}_2\mathcal{A}_2^*[\eta_{\alpha_k}]\rangle = \langle \widetilde{\xi}, \xi\rangle - \lim_{k\to\infty}\sqrt{\alpha_k}\,\langle \widetilde{\xi}, \sqrt{\alpha_k}\eta_{\alpha_k}\rangle\\ &= \langle \widetilde{\xi}, \xi\rangle - \langle \widetilde{\xi}, \xi_0\rangle\lim_{k\to\infty}\sqrt{\alpha_k} = \langle \widetilde{\xi}, \xi\rangle\,,\end{aligned}$$

implying $\xi = \mathcal{A}_2\psi_2$. However, this contradicts our assumption that $\xi \notin \operatorname{Ran}\mathcal{A}_2$ hence proving (A.2).

Now we calculate

$$\|\mathcal{A}_2^*[\xi_\alpha]\| = \frac{1}{\langle \xi, \eta_\alpha\rangle^{3/4}}\|\mathcal{A}_2^*[\eta_\alpha]\| \le \frac{1}{\langle \xi, \eta_\alpha\rangle^{3/4}}\langle \xi, \eta_\alpha\rangle^{1/2} = \langle \xi, \eta_\alpha\rangle^{-1/4}\,,$$

which converges to 0 as $\alpha \to 0+$ due to (A.2). Here we also use an upper bound for $\|\mathcal{A}_2^*[\eta_\alpha]\|$ from (A.1). Now to show $\lim_{\alpha\to 0+}\|\mathcal{A}_1^*[\xi_\alpha]\| = \infty$, we recall that $\xi \in \operatorname{Ran}\mathcal{A}_1$, i.e., $\xi = \mathcal{A}_1[\psi_1]$ for some $\psi_1 \in Y_1$. Employing the Cauchy-Schwarz inequality, we next observe that

$$\|\mathcal{A}_1^*[\xi_\alpha]\| \ge \frac{\langle \mathcal{A}_1^*[\xi_\alpha], \psi_1\rangle}{\|\psi_1\|} = \frac{\langle \xi, \xi_\alpha\rangle}{\|\psi_1\|} = \frac{\langle \xi, \eta_\alpha\rangle^{1/4}}{\|\psi_1\|}\,,$$

which readily gives $\lim_{\alpha\to 0+}\|\mathcal{A}_1^*[\xi_\alpha]\| = \infty$ in consideration of (A.2). □

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