

Inverse Problems



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Far field operator splitting by principal component pursuit

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Abstract

We consider scattering of time-harmonic plane waves by an ensemble of well-separated compactly supported inhomogeneous scatterers. The far field operator, which maps superpositions of plane wave incident fields to the far field patterns of the associated scattered fields, is commonly used as an idealized description of data sets obtained in corresponding remote sensing experiments. Suppose that some *a priori* information about the approximate position of just one of the scatterers in the ensemble is available, and that the distance between this scatterer and the others is sufficiently large. This article is about recovering the far field operator associated to this single scatterer from the far field operator associated to the whole collection of scatterers. Due to multiple scattering effects this is a nonlinear inverse problem. Assuming that these multiple scattering effects are not too strong, we show that an approximate solution can be obtained by decomposing the far field operator into a sparse component and a low-rank component, and we apply a convex program called principal component pursuit for this purpose. We give necessary and sufficient conditions for unique solvability, establish a stability result and provide numerical examples to illustrate our theoretical findings.

1. Introduction

This article is concerned with far field operators for the scattering of scalar time-harmonic plane waves by compactly supported inhomogeneous obstacles. These operators are defined in terms of the far field patterns of scattered fields corresponding to plane wave incident fields at a single frequency for all possible observation and incident directions. Far field operators map densities of superpositions of such plane waves to the far field patterns of the associated scattered fields. They are a popular model for remote sensing observations in inverse scattering theory and their properties have been widely studied in the literature. It is, for instance, well known that the refractive index of a bounded penetrable scattering object is uniquely determined by the associated far field operator (see [4, 36, 38, 40]). Accordingly, far field operators are at the foundation of many successful reconstruction methods for the inverse medium scattering problem (see, e.g. [1, 10, 11, 14, 21, 30] and the monographs [5, 6, 13, 33]).

We shall study a particular data splitting problem for far field operators corresponding to ensembles of finitely many well-separated penetrable scatterers. Well-separated means here that the diameters of the supports of the scatterers in the ensemble are small compared to their distances to each other. Assuming that the approximate locations of all scatterers in such an ensemble are available *a priori*, it has recently been shown in [22] that approximations of the far field operators associated to each scatterer individually can be recovered given the far field operator associated to the whole collection of scatterers. Due to multiple scattering between the different scatterers in the ensemble, which has to be disentangled and removed by the splitting method, this is a nonlinear inverse problem. The method developed in [22] is based on a careful analysis of the second order Born approximation of the scattering problem, which is valid when multiple scattering is not too strong. The associated algorithms crucially rely on sparse representations of the far field operators corresponding to the individual scatterers in the ensemble, which

are determined by the *a priori* information on their approximate locations. Using these sparse representations, either least squares or a convex program called basis pursuit (see, e.g. [9, 17, 18]) has been applied to recover the unknown far field operator components in [22].

In this work we aim to significantly reduce the required *a priori* information compared to [22]. We ask whether it is possible to extract the far field operator associated to a single scatterer in an ensemble of well-separated compactly supported scatterers from the far field operator associated to the whole ensemble, assuming that just the approximate location of this single scatterer is known, but nothing about the other scatterers. In other words, our goal is to isolate or split off the information about a single scatterer contained in the far field operator for the whole ensemble. For instance, this is relevant if one aims to focus on recovering information about a single scatterer within a collection of scatterers or about scatterers inside a certain region of interest. As in [22], the *a priori* information on the position of the single scatterer allows for a sparse representation of the associated far field operator component that we aim to recover. Moreover, we will show that the remaining part of the far field operator can be approximated by a low-rank operator, which however is not sparse. Accordingly, the inverse problem is equivalent to recovering a sparse operator given the sum of this sparse operator with a low-rank operator plus some modeling error and possibly some data error. We show that a convex program called principal component pursuit (PCP) (see [7, 8, 45]) can be applied for this purpose.

The main theoretical result of this work is a stability estimate for the solution of this sparse plus low-rank far field operator splitting obtained by PCP. This stability estimate bounds the norm of the error in the reconstruction of the far field operator associated to the single scatterer in terms of a constant, which is determined by the geometric setup of the whole ensemble of scatterers and the wave length, times the norm of the sum of all modeling and data errors. In order to achieve this result, we first have to analyze an idealized version of the sparse plus low-rank splitting problem without taking into account modeling and data errors. We establish a sufficient optimality condition for the associated optimization problem, which is then used in the proof of the stability result. Our analysis builds on corresponding results for robust sparse plus low-rank matrix splitting from [7, 8, 45] and on sparsity and low-rank properties of far field operators established in [22].

Previous investigations of far field splitting based on suitable sparse representations of far field patterns have mainly been concentrated on source problems or scattering problems with a single incident wave. However, the basic reasoning developed in [20, 23–26] is closely related to the perspective taken in the present work. Other approaches to wave splitting for time-harmonic inverse source problems have been proposed in [3, 39]. Furthermore, splitting problems for time-dependent scattering problems have been considered in [2, 19, 28].

The remainder of this article proceeds as follows. In section 2 we briefly outline the theoretical background of the scattering problem and discuss sparsity and low-rank properties of far field operators. Then, in section 3, we describe how PCP can be used to approximate solutions to the splitting problem considered in the paper, and we analyze the stability of this approach. Numerical results are given in section 4, and we finally close with some conclusions.

2. Inhomogeneous medium scattering

Before we discuss sparsity and low-rank properties of far field operators associated to ensembles of compactly supported scatterers, we summarize some facts and basic notations concerning the direct scattering problem. We consider time-harmonic scattering from penetrable scatterers modeled by the Helmholtz equation in $\mathbb{R}^2$. Let $k > 0$ denote the wave number and $n^2 = 1 + q$ the index of refraction for a real-valued compactly supported contrast function $q \in L^\infty(\mathbb{R}^2)$ satisfying $q > -1$ a.e. in $\mathbb{R}^2$ and $q = 0$ a.e. in $\mathbb{R}^2 \setminus D$ for some bounded open subset $D \subseteq \mathbb{R}^2$. We call D the support of the scatterers and below we will assume D to be the union of a few well-separated connected components.

Suppose the scatterers are illuminated by an incident plane wave

$$u^i(\mathbf{x}; \mathbf{d}) := e^{ik\mathbf{x} \cdot \mathbf{d}}, \quad \mathbf{x} \in \mathbb{R}^2, \quad (2.1a)$$

along the illumination direction $\mathbf{d} \in S^1$. Then the total field $u_q \in H_{\text{loc}}^1(\mathbb{R}^2)$ and the associated scattered field $u_q^s = u_q - u^i$ solve the Helmholtz equation

$$\Delta u_q + k^2 n^2 u_q = 0 \quad \text{in } \mathbb{R}^2, \quad (2.1b)$$

together with the Sommerfeld radiation condition

$$\lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial u_q^s}{\partial r}(\mathbf{x}; \mathbf{d}) - iku_q^s(\mathbf{x}; \mathbf{d}) \right) = 0, \quad r = |\mathbf{x}| \rightarrow \infty, \quad (2.1c)$$

uniformly with respect to all directions $\mathbf{x}/|\mathbf{x}| \in S^1$. We often indicate dependencies on the illumination direction $\mathbf{d}$ by a second argument and dependencies on the contrast q by a subscript.

The unique weak solution $u_q \in H_{\text{loc}}^1(\mathbb{R}^2)$ of (2.1) (see, e.g. [32, theorem 7.13]) exhibits an asymptotic expansion as an outgoing cylindrical wave

$$u_q^s(\mathbf{x}; \mathbf{d}) = \frac{e^{i\pi/4}}{\sqrt{8\pi}} \frac{e^{ik|\mathbf{x}|}}{\sqrt{k|\mathbf{x}|}} u_q^\infty(\hat{\mathbf{x}}; \mathbf{d}) + O(|\mathbf{x}|^{-\frac{3}{2}}), \quad |\mathbf{x}| \rightarrow \infty,$$

uniformly in all observation directions $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}| \in S^1$ (see, e.g. [32, theorem 7.15]). Here, the far field pattern $u_q^\infty \in L^2(S^1 \times S^1)$ is given by

$$u_q^\infty(\hat{\mathbf{x}}; \mathbf{d}) = k^2 \int_D q(\mathbf{y}) u_q(\mathbf{y}; \mathbf{d}) e^{-ik\hat{\mathbf{x}} \cdot \mathbf{y}} d\mathbf{y}, \quad \hat{\mathbf{x}} \in S^1, \quad (2.2)$$

(see, e.g. [32, theorem 7.15]).

Accordingly, the far field operator

$$F_q : L^2(S^1) \rightarrow L^2(S^1), \quad (F_q g)(\hat{\mathbf{x}}) := \int_{S^1} g(\mathbf{d}) u_q^\infty(\hat{\mathbf{x}}; \mathbf{d}) d\mathbf{d}, \quad (2.3)$$

which is determined by the far field patterns $u_q^\infty(\hat{\mathbf{x}}; \mathbf{d})$ for all possible observation and incident directions $\hat{\mathbf{x}}, \mathbf{d} \in S^1$, is often considered as an idealized measurement operator for remote sensing experiments in inverse scattering theory (see, e.g. [13]). This operator is well-known to be compact and normal (see, e.g. [32, theorem 7.20]). Moreover, it is of trace class (see [12]) and thus in particular a Hilbert–Schmidt operator. We refer, e.g. to [41, section VI.6] for a brief introduction to this class of operators. For later reference we denote the space of Hilbert–Schmidt operators on $L^2(S^1)$ by $\text{HS}(L^2(S^1))$.

2.1. Far field operator splitting

To simplify the presentation we restrict the discussion in the following to ensembles of two scatterers. Accordingly, we assume that the scatterer $D = D_1 \cup D_2$ consists of two well-separated components $D_1, D_2 \subseteq \mathbb{R}^2$ such that $D_j \subseteq B_{R_j}(\mathbf{c}_j)$, $j = 1, 2$, for some $R_1, R_2 > 0$ and $\mathbf{c}_1, \mathbf{c}_2 \in \mathbb{R}^2$ satisfying $|\mathbf{c}_1 - \mathbf{c}_2| \gg R_1 + R_2$, and we denote $q_1 := q|_{D_1}$ and $q_2 := q|_{D_2}$. However, we note that our analysis immediately extends to the case when $D \setminus D_1$ consists of more than one component.

Let us first recap the far field operator splitting problem as considered in [22]. Given the far field operator F_q , the wave number k and sufficient *a priori* information on the location of D_1 and D_2 (i.e. about $\mathbf{c}_1$ and $\mathbf{c}_2$) the goal in [22] was to recover the far field operators F_{q_1} and F_{q_2} corresponding to the two individual components of the scatterer. The procedure in [22] is based on splitting the given data into three parts,

$$F_q = F_{q_1} + F_{q_2} + F_{q_1, q_2},$$

and approximating each of these three components by sparse operators with respect to certain suitably modulated Fourier bases of $L^2(S^1)$. In this decomposition $F_{q_1, q_2} = F_q - (F_{q_1} + F_{q_2})$ represents those multiple scattering components that involve scattering both on q_1 and q_2 and therefore can neither be assigned to F_{q_1} nor to F_{q_2} . Accordingly, F_{q_1, q_2} is the part of F_q that has to be removed when recovering F_{q_1} and F_{q_2} .

In this paper, we consider a modified version of this far field operator splitting problem. We assume that only *a priori* information on the approximate location of D_1 (i.e. about $\mathbf{c}_1$) is available, and we are only interested in recovering the far field operator F_{q_1} associated to the first component of the scatterer. Accordingly, we split the given far field data

$$F_q = F_{q_1} + F_{q \setminus q_1}$$

into two parts. Our strategy remains to approximate F_{q_1} by a sparse operator with respect to the same modulated Fourier basis as in [22]. However, in contrast to [22], we now approximate the remaining part $F_{q \setminus q_1}$ by an operator of low rank, which does not require any *a priori* information on the approximate location of D_2 (i.e. about $\mathbf{c}_2$).

2.2. Sparse and low-rank approximations of far field operators

If $u_q \in H_{\text{loc}}^1(\mathbb{R}^2)$ is the solution to the scattering problem (2.1), then $u_q|_D \in L^2(D)$ satisfies the Lippmann–Schwinger integral equation

$$u_q(\mathbf{x}; \mathbf{d}) = u^i(\mathbf{x}; \mathbf{d}) + k^2 \int_{\mathbb{R}^2} q(\mathbf{y}) u_q(\mathbf{y}; \mathbf{d}) \Phi(\mathbf{x} - \mathbf{y}) \, \mathbf{y}, \quad \mathbf{x} \in D,$$

(cf e.g. [32, theorem 7.12]). Here, $\Phi(\mathbf{x}) := \frac{i}{4} H_0^{(1)}(k|\mathbf{x}|)$, $\mathbf{x} \neq \mathbf{0}$, denotes the fundamental solution to the Helmholtz equation. Introducing $L_q : L^2(D) \rightarrow L^2(D)$ by

$$(L_q f)(\mathbf{x}) := k^2 \int_D q(\mathbf{y}) f(\mathbf{y}) \Phi(\mathbf{x} - \mathbf{y}) \, \mathbf{y}, \quad \mathbf{x} \in D,$$

yields the Born series representation

$$u_q^s(\cdot; \mathbf{d}) = \sum_{l=1}^{\infty} (L_q)^l u^i(\cdot; \mathbf{d}) \quad \text{in } \mathbb{R}^2, \quad (2.4)$$

of the scattered field, provided that the operator norm $\|L_q\|$ is strictly less than one (see, e.g. [29, 31, 37, 42, 43] for more detailed discussions). Neglecting terms of order $l \geq 3$ in (2.4) (i.e. considering the second order Born approximation) and substituting the result into (2.2) and (2.3), the working hypothesis of [22] was that the far field operator can be approximated in terms of

$$F_q \approx F_1 + F_2 + (F_{1,2} + F_{2,1}), \quad (2.5)$$

where $F_1 \approx F_{q_1}$ and $F_2 \approx F_{q_2}$ denote the second order Born approximations of the far field operators F_{q_1} and F_{q_2} associated to the two components of the scatterer in D_1 and D_2 , respectively, and $F_{1,2} + F_{2,1} \approx F_q - F_{q_1} - F_{q_2}$ is the second order Born approximation of the remaining multiple scattering effects. Also in the present article we assume this premise to be valid, i.e. to be working in a suitable weak scattering regime. The operators $F_1, F_2, F_{1,2}$, and $F_{2,1}$ are defined analogously to (2.3) with $u_q^\infty(\hat{\mathbf{x}}, \mathbf{d})$, $\hat{\mathbf{x}}, \mathbf{d} \in S^1$, replaced by

$$\begin{aligned} u_1^\infty(\hat{\mathbf{x}}; \mathbf{d}) &:= k^2 \int_{D_1} q_1(\mathbf{y}) e^{ik(\mathbf{d} - \hat{\mathbf{x}}) \cdot \mathbf{y}} \, \mathbf{y} \\ &\quad + k^4 \int_{D_1} \int_{D_1} q_1(\mathbf{y}_2) q_1(\mathbf{y}_1) \Phi(\mathbf{y}_2 - \mathbf{y}_1) e^{ik(\mathbf{d} \cdot \mathbf{y}_1 - \hat{\mathbf{x}} \cdot \mathbf{y}_2)} \, \mathbf{y}_1 \, \mathbf{y}_2, \end{aligned} \quad (2.6a)$$

$$\begin{aligned} u_2^\infty(\hat{\mathbf{x}}; \mathbf{d}) &:= k^2 \int_{D_2} q_2(\mathbf{y}) e^{ik(\mathbf{d} - \hat{\mathbf{x}}) \cdot \mathbf{y}} \, \mathbf{y} \\ &\quad + k^4 \int_{D_2} \int_{D_2} q_2(\mathbf{y}_2) q_2(\mathbf{y}_1) \Phi(\mathbf{y}_2 - \mathbf{y}_1) e^{ik(\mathbf{d} \cdot \mathbf{y}_1 - \hat{\mathbf{x}} \cdot \mathbf{y}_2)} \, \mathbf{y}_1 \, \mathbf{y}_2, \end{aligned} \quad (2.6b)$$

$$u_{1,2}^\infty(\hat{\mathbf{x}}; \mathbf{d}) := k^4 \int_{D_1} \int_{D_2} q_1(\mathbf{y}_2) q_2(\mathbf{y}_1) \Phi(\mathbf{y}_2 - \mathbf{y}_1) e^{ik(\mathbf{d} \cdot \mathbf{y}_1 - \hat{\mathbf{x}} \cdot \mathbf{y}_2)} \, \mathbf{y}_1 \, \mathbf{y}_2, \quad (2.6c)$$

and

$$u_{2,1}^\infty(\hat{\mathbf{x}}; \mathbf{d}) := k^4 \int_{D_2} \int_{D_1} q_2(\mathbf{y}_2) q_1(\mathbf{y}_1) \Phi(\mathbf{y}_2 - \mathbf{y}_1) e^{ik(\mathbf{d} \cdot \mathbf{y}_1 - \hat{\mathbf{x}} \cdot \mathbf{y}_2)} \, \mathbf{y}_1 \, \mathbf{y}_2, \quad (2.6d)$$

respectively.

For $j = 1, 2$, we introduce the far field translation operator

$$T_{\mathbf{c}_j} : L^2(S^1) \rightarrow L^2(S^1), \quad (T_{\mathbf{c}_j} g)(\hat{\mathbf{x}}) := e^{ik\mathbf{c}_j \cdot \hat{\mathbf{x}}} g(\hat{\mathbf{x}}). \quad (2.7)$$

Denoting by $(\mathbf{e}_n)_n := (e^{in \arg(\cdot)} / \sqrt{2\pi})_n$ the standard Fourier basis of $L^2(S^1)$, we define the finite dimensional subspaces $\mathcal{V}_{N_j, N_l}^{\mathbf{c}_j, \mathbf{c}_l} \subseteq \text{HS}(L^2(S^1))$ for all possible combinations of $j, l \in \{1, 2\}$ by

$$\mathcal{V}_{N_j, N_l}^{\mathbf{c}_j, \mathbf{c}_l} := \left\{ G \in \text{HS}(L^2(S^1)) \mid Gg = \sum_{|m| \leq N_j} \sum_{|n| \leq N_l} a_{m,n} T_{-\mathbf{c}_j} \mathbf{e}_m \langle g, T_{-\mathbf{c}_l} \mathbf{e}_n \rangle_{L^2(S^1)}, a_{m,n} \in \mathbb{C} \right\}. \quad (2.8)$$

We call $(T_{-\mathbf{c}_j} \mathbf{e}_m)_m$ and $(T_{-\mathbf{c}_l} \mathbf{e}_n)_n$ modulated Fourier bases of $L^2(S^1)$, the expansion in (2.8) a modulated Fourier series expansion of G , and the coefficients $(a_{m,n})_{m,n}$ modulated Fourier coefficients of G . We also

use the short hand $\mathcal{V}_{N_j}^{c_j} := \mathcal{V}_{N_j, N_j}^{c_j}$, $j = 1, 2$. One of the main outcomes of [22] has been that the far field operator components F_1 , F_2 , $F_{1,2}$, and $F_{2,1}$ in the second order Born approximation (2.5) of the far field operator F_q can be well approximated in $\mathcal{V}_{N_1}^{c_1}$, $\mathcal{V}_{N_2}^{c_2}$, $\mathcal{V}_{N_1, N_2}^{c_1, c_2}$, and $\mathcal{V}_{N_2, N_1}^{c_2, c_1}$, respectively, whenever we choose $N_1 \gtrsim kR_1$ and $N_2 \gtrsim kR_2$. Here the symbol $\gtrsim$ means that the left hand side is larger or equal to γ times the right hand side for some constant factor $\gamma > 1$. In practice we typically use $\gamma = e/2$. Corresponding approximation error estimates have been provided in [22, lemmas 2.4 and 3.4]. The basic idea behind the translation operators T_{-c_j} and T_{-c_l} in (2.8) is to counteract the oscillatory behavior of the integrands in (2.6) as much as possible in order to be able to approximate the far fields u_1^∞ , u_2^∞ , $u_{1,2}^\infty$, and $u_{2,1}^\infty$ by modulated Fourier series expansions of low order.

For moderate values of $N_1 \in \mathbb{N}$ the approximation of F_1 in $\mathcal{V}_{N_1}^{c_1}$ as in (2.8) can be considered to be sparse, because it involves at most $(2N_1 + 1)^2$ nonzero coefficients. Defining another finite dimensional subspace $\mathcal{W}_{N_2}^{c_2} \subseteq \text{HS}(L^2(S^1))$ by

$$\mathcal{W}_{N_2}^{c_2} := \left\{ G \in \text{HS}(L^2(S^1)) \mid Gg = \sum_{|n| \leq N_2} \left(T_{-c_2} \mathbf{e}_n \langle g, \alpha_n \rangle_{L^2(S^1)} + \beta_n \langle g, T_{-c_2} \mathbf{e}_n \rangle_{L^2(S^1)} \right), \alpha_n, \beta_n \in L^2(S^1) \right\}, \quad (2.9)$$

we immediately observe that $\mathcal{V}_{N_j, N_l}^{c_j, c_l} \subseteq \mathcal{W}_{N_2}^{c_2}$ whenever $j = 2$ or $l = 2$. Consequently, the sum of operators $F_2 + F_{1,2} + F_{2,1}$ can be well approximated in $\mathcal{W}_{N_2}^{c_2}$. Since

$$\text{rank} G \leq 2(2N_2 + 1) \quad \text{for any } G \in \mathcal{W}_{N_2}^{c_2},$$

this implies that for moderate values of $N_2 \in \mathbb{N}$ the approximation of $F_2 + F_{1,2} + F_{2,1}$ in $\mathcal{W}_{N_2}^{c_2}$ as in (2.9) is low-rank (at least its rank is finite and not too large).

To sum up, the idea in the following is to compute a sparse operator $F_1 \in \mathcal{V}_{N_1}^{c_1}$ and a low-rank operator $L \in \mathcal{W}_{N_2}^{c_2}$ such that

$$F_q \approx F_1 + L \quad (2.10)$$

and to approximate F_{q_1} by F_1 and $F_{q \setminus q_1} = F_q - F_{q_1}$ by L .

Example 2.1. We illustrate the essential supports of the modulated Fourier coefficients of F_{q_1} and $F_{q \setminus q_1}$ by a numerical example with $q = \chi_{D_1} - 0.5\chi_{D_2}$ for D_1 and D_2 as shown in figure 1 (left). Using a Nyström method (see, e.g. [13, pp 91–96]) for the associated transmission problem, we simulate the far field patterns $u_q^\infty(\hat{\mathbf{x}}_m; \mathbf{d}_n)$ and $u_{q_1}^\infty(\hat{\mathbf{x}}_m; \mathbf{d}_n)$ for $M = 170$ equidistant observation and illumination directions on S^1 at wave number $k = 1$. The matrices

$$F_{q_1} := \frac{2\pi}{M} [u_{q_1}^\infty(\hat{\mathbf{x}}_m; \mathbf{d}_n)]_{1 \leq m, n \leq M} \quad \text{and} \quad F_{q \setminus q_1} := \frac{2\pi}{M} [u_q^\infty(\hat{\mathbf{x}}_m; \mathbf{d}_n) - u_{q_1}^\infty(\hat{\mathbf{x}}_m; \mathbf{d}_n)]_{1 \leq m, n \leq M}$$

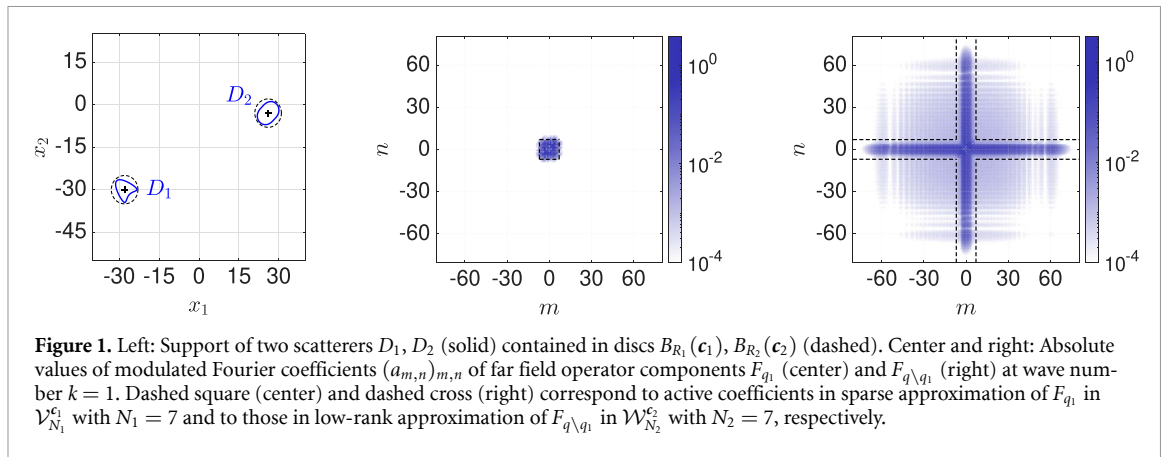
then approximate the far field operator components F_{q_1} and $F_{q \setminus q_1}$. The dashed discs in figure 1 (left) are centered at $\mathbf{c}_1 = (-28, -30)^\top$ and $\mathbf{c}_2 = (26, -3)^\top$ with radii $R_1 = R_2 = 5$, respectively. Taking the two-dimensional Fourier transform of F_{q_1} and $F_{q \setminus q_1}$ after multiplication with the appropriate modulation factors yields an approximation of the modulated Fourier coefficients $(a_{m,n})_{m,n}$ of F_{q_1} in terms of $(T_{-c_1} \mathbf{e}_n)_n$ and of $F_{q \setminus q_1}$ in terms of $(T_{-c_2} \mathbf{e}_n)_n$, respectively. In figure 1(center and right) the absolute values of these modulated Fourier coefficients are plotted using a logarithmic color scale.

In the plot in the center, it can be observed that they are essentially supported in a square $[-N_1, N_1]^2$ for $N_1 = 7 \gtrsim 5 = kR_1$. This illustrates that, as predicted by the theoretical results from [22] mentioned above, the far field operator F_{q_1} can be well approximated in $\mathcal{V}_{N_1}^{c_1}$ as introduced in (2.8).

In the right plot, the coefficients with the largest magnitude are essentially supported in the cross-shaped index set $\{(m, n) : |m| \leq N_2 \text{ or } |n| \leq N_2\}$ for $N_2 = 7 \gtrsim 5 = kR_2$. This illustrates that, as predicted by our theoretical results, the far field operator $F_{q \setminus q_1}$ can be well approximated in $\mathcal{W}_{N_2}^{c_2}$ as introduced in (2.9).

We note however that the contribution of the coefficients outside the cross-shaped index set in the plot on the right is larger than outside the square in the plot in the center, which leads to a larger approximation error for the projection of $F_{q \setminus q_1}$ onto $\mathcal{W}_{N_2}^{c_2}$ compared to the projection of F_{q_1} onto $\mathcal{V}_{N_1}^{c_1}$. This is caused by those far field operator components of scattering order three or higher, that can actually be approximated well in $\mathcal{V}_{N_1}^{c_1}$, but belong to $F_{q \setminus q_1}$ and not to F_{q_1} . $\diamond$

The splitting ansatz in (2.10) will only work reliably if we can ensure that the low-rank operators in $\mathcal{W}_{N_2}^{c_2}$ do not have a sparse representation in $\mathcal{V}_{N_1}^{c_1}$. The following section addresses this question.



2.3. When does sparse plus low-rank far field operator splitting work?

To avoid cluttered notation we write $\mathcal{V} := \mathcal{V}_{N_1}^{c_1}$ and $\mathcal{W} := \mathcal{W}_{N_2}^{c_2}$ in the following. We denote the orthogonal projections onto $\mathcal{V}$ and $\mathcal{W}$ by $\mathcal{P}_{\mathcal{V}}$ and $\mathcal{P}_{\mathcal{W}}$, respectively, and we define $\mathcal{P}_{\mathcal{V}^\perp} := \mathcal{I} - \mathcal{P}_{\mathcal{V}}$ and $\mathcal{P}_{\mathcal{W}^\perp} := \mathcal{I} - \mathcal{P}_{\mathcal{W}}$.

We start with a general observation regarding the splitting problem under consideration. Operators $G = G_1 + G_2 \in \mathcal{V} + \mathcal{W}$ can be uniquely decomposed into $G_1 \in \mathcal{V}$ and $G_2 \in \mathcal{W}$ if and only if $\mathcal{V} \cap \mathcal{W} = \{0\}$. This is the case if and only if the cosine of the minimal angle between $\mathcal{V}$ and $\mathcal{W}$,

$$\|\mathcal{P}_{\mathcal{V}}\mathcal{P}_{\mathcal{W}}\| = \sup_{H \in \mathcal{V}, K \in \mathcal{W}} \frac{|\langle H, K \rangle_{\text{HS}}|}{\|H\|_{\text{HS}}\|K\|_{\text{HS}}}, \quad (2.11)$$

is strictly smaller than one (cf e.g. [15, lemma 2.10]). Moreover, the smaller $\|\mathcal{P}_{\mathcal{V}}\mathcal{P}_{\mathcal{W}}\|$ is, the more robust splitting becomes with respect to noise in the data G . In fact it is straightforward to check that in this case

$$\|G_j\|_{\text{HS}}^2 \leq \frac{1}{1 - \|\mathcal{P}_{\mathcal{V}}\mathcal{P}_{\mathcal{W}}\|^2} \|G\|_{\text{HS}}^2, \quad j = 1, 2.$$

Thus, the aim of this section is to develop an upper bound for $\|\mathcal{P}_{\mathcal{V}}\mathcal{P}_{\mathcal{W}}\|$, similar to what has been established in [22, propositions 3.8 and 3.14]. We start with some additional notations and technical remarks.

Remark 2.2. Apart from the usual Hilbert–Schmidt norm, we will also use some other norms on $\text{HS}(L^2(S^1))$ in our analysis below. We first recall that any operator $G \in \text{HS}(L^2(S^1))$ can be expanded in terms of the Fourier basis $(e_n)_n \subseteq L^2(S^1)$ via

$$Gf = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} a_{m,n} e_m \langle f, e_n \rangle_{L^2(S^1)}, \quad g \in L^2(S^1), \quad (2.12)$$

with Fourier coefficients $(a_{m,n})_{m,n} = (\langle G e_n, e_m \rangle_{L^2(S^1)})_{m,n} \in \ell^2 \times \ell^2$. For $1 \leq p \leq \infty$ this representation gives rise to the norms

$$\|G\|_{\ell^p \times \ell^p} := \|(a_{m,n})_{m,n}\|_{\ell^p \times \ell^p}, \quad G \in \text{HS}(L^2(S^1)),$$

at least if the right hand side is finite. On the other hand, $G \in \text{HS}(L^2(S^1))$ also possesses a singular value decomposition

$$Gf = \sum_{n \in \mathbb{N}} \sigma_n u_n \langle f, v_n \rangle_{L^2(S^1)}, \quad f \in L^2(S^1),$$

with singular values $(\sigma_n)_n \in \ell^2$ and singular vectors $(u_n)_n, (v_n)_n \subseteq L^2(S^1)$. Therewith the Hilbert–Schmidt norm of an operator G can be written as $\|G\|_{\text{HS}} = \|(\sigma_n)_n\|_{\ell^2}$, its nuclear norm as $\|G\|_{\text{nuc}} = \|(\sigma_n)_n\|_{\ell^1}$, and its operator norm as $\|G\| = \|(\sigma_n)_n\|_{\ell^\infty}$. Furthermore, Parseval’s identity shows that

$$\langle G, H \rangle_{\text{HS}} = \langle G, H \rangle_{\ell^2 \times \ell^2} \quad \text{for } G, H \in \text{HS}(L^2(S^1)).$$

We also denote the number of nonzero coefficients $(a_{m,n})_{m,n}$ in (2.12) by $\|G\|_{\ell^0 \times \ell^0}$ and we recall that the number of nonzero singular values $(\sigma_n)_n$ of G coincides with $\text{rank}(G)$. $\diamond$

To relate the modulated Fourier bases in the definitions (2.8) of $\mathcal{V}$ and (2.9) of $\mathcal{W}$ to the unmodulated Fourier basis $(\mathbf{e}_n)_n$, we introduce for any $\mathbf{c} \in \mathbb{R}^2$ the operator

$$\mathcal{T}_{\mathbf{c}} : \text{HS}(L^2(S^1)) \rightarrow \text{HS}(L^2(S^1)), \quad \mathcal{T}_{\mathbf{c}} G := T_{\mathbf{c}} \circ G \circ T_{-\mathbf{c}}, \quad (2.13)$$

where $T_{\mathbf{c}}$ has already been defined in (2.7). We recall from [22, section 2.2] that $F_{q(\cdot+\mathbf{c})} = \mathcal{T}_{\mathbf{c}} F_q$, i.e. the operator $\mathcal{T}_{\mathbf{c}}$ describes the transformation of far field operators under translation of the associated scatterer by $\mathbf{c}$. Similarly, it follows immediately from (2.8) and (2.9) that the subspaces $\mathcal{V}$ and $\mathcal{W}$ can be rewritten as

$$\mathcal{V} = \{G \in \text{HS}(L^2(S^1)) \mid \mathcal{T}_{\mathbf{c}_1} G \in \mathcal{V}_{N_1}^0\} \quad \text{and} \quad \mathcal{W} = \{G \in \text{HS}(L^2(S^1)) \mid \mathcal{T}_{\mathbf{c}_2} G \in \mathcal{W}_{N_2}^0\}.$$

The mapping properties of $\mathcal{T}_{\mathbf{c}}$ have proven to be an essential tool in [22]. The following lemma presents further results that are relevant for this work.

Lemma 2.3. *Let $\mathbf{c} \in \mathbb{R}^2$. Then,*

- (a) $\mathcal{T}_{\mathbf{c}}$ is a unitary operator on $\text{HS}(L^2(S^1))$ with $\mathcal{T}_{\mathbf{c}}^* = \mathcal{T}_{-\mathbf{c}}$,
- (b) $\mathcal{T}_{\mathbf{c}}$ preserves the norms $\|\cdot\|_{\text{HS}}$, $\|\cdot\|_{\text{nuc}}$ and $\|\cdot\|$ as well as the rank, and
- (c) for any $N \in \mathbb{N}$, we have that

$$\|\mathcal{T}_{\mathbf{c}} G\|_{\ell^\infty \times \ell^\infty} \leq 2\beta_N^{\mathbf{c}} \|G\|_{\text{HS}}, \quad G \in \mathcal{W}_N^0, \quad (2.14)$$

where

$$\beta_N^{\mathbf{c}} := \sqrt{\sup_{n \in \mathbb{Z}} \left(\sum_{|n'-n| \leq N} J_{n'}^2(k|\mathbf{c}|) \right)} \leq \min \left\{ 1, b \sqrt{\frac{2N+1}{k|\mathbf{c}|}} \right\} \quad (2.15)$$

with $b \approx 0.7595$.

Proof. Part (a) follows immediately from the definitions (2.7) and (2.13) of $T_{\mathbf{c}}$ and $\mathcal{T}_{\mathbf{c}}$ (see [22, lemma 2.2]).

Since applying the unitary operator $\mathcal{T}_{\mathbf{c}}$ does not change the singular values of an operator $G \in \text{HS}(L^2(S^1))$, part (b) follows from the characterizations of $\|\cdot\|_{\text{HS}}$, $\|\cdot\|_{\text{nuc}}$ and $\|\cdot\|$ in terms of the singular values at the end of remark 2.2.

Part (c) requires a bit more effort. By definition (2.9) of $\mathcal{W}_N^0$ and since

$$\begin{aligned} & \{(m', n') \mid |m'| \leq N \text{ or } |n'| \leq N\} \\ &= \{(m', n') \mid m' \in \mathbb{Z}, |n'| \leq N\} \cup \{(m', n') \mid |m'| \leq N, |n'| > N\}, \end{aligned}$$

we can write any $G \in \mathcal{W}_N^0$ as

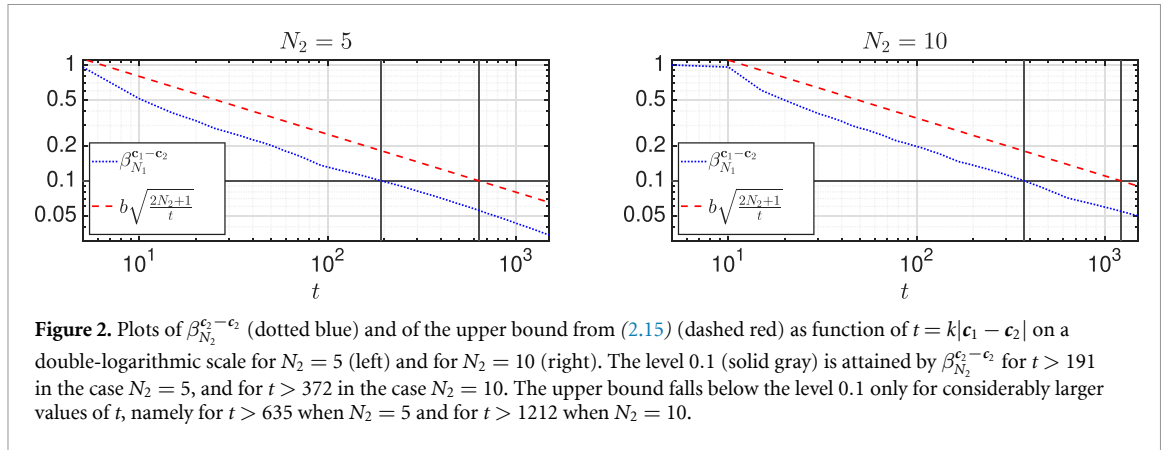
$$G = \sum_{m' \in \mathbb{Z}} \sum_{|n'| \leq N} a_{m', n'} \mathbf{e}_{m'} \langle \cdot, \mathbf{e}_{n'} \rangle_{L^2(S^1)} + \sum_{|m'| \leq N} \sum_{|n'| > N} a_{m', n'} \mathbf{e}_{m'} \langle \cdot, \mathbf{e}_{n'} \rangle_{L^2(S^1)}.$$

We conclude by expanding $T_{\mathbf{c}} \mathbf{e}_{m'}$ in terms of $(\mathbf{e}_m)_m$ and $T_{\mathbf{c}} \mathbf{e}_{n'}$ in terms of $(\mathbf{e}_n)_n$ that the coefficients $(a_{m, n}^{\mathbf{c}})_{m, n}$ in the Fourier expansion of $\mathcal{T}_{\mathbf{c}} G$ as in (2.12) are given by

$$\begin{aligned} a_{m, n}^{\mathbf{c}} &= \sum_{m' \in \mathbb{Z}} \sum_{|n'| \leq N} a_{m', n'} \langle T_{\mathbf{c}} \mathbf{e}_{m'}, \mathbf{e}_m \rangle_{L^2(S^1)} \langle \mathbf{e}_n, T_{\mathbf{c}} \mathbf{e}_{n'} \rangle_{L^2(S^1)} \\ &\quad + \sum_{|m'| \leq N} \sum_{|n'| > N} a_{m', n'} \langle T_{\mathbf{c}} \mathbf{e}_{m'}, \mathbf{e}_m \rangle_{L^2(S^1)} \langle \mathbf{e}_n, T_{\mathbf{c}} \mathbf{e}_{n'} \rangle_{L^2(S^1)} \end{aligned}$$

for $m, n \in \mathbb{Z}$. Throughout this work, $\langle \cdot, \cdot \rangle_{L^2(S^1)}$ is linear in its first and antilinear in its second argument. Accordingly, applying the Cauchy–Schwarz inequality we deduce that

$$\begin{aligned} \|\mathcal{T}_{\mathbf{c}} G\|_{\ell^\infty \times \ell^\infty} &= \|(a_{m, n}^{\mathbf{c}})_{m, n}\|_{\ell^\infty \times \ell^\infty} \\ &\leq 2 \sqrt{\sup_{m \in \mathbb{Z}} \left(\sum_{m' \in \mathbb{Z}} |\langle T_{\mathbf{c}} \mathbf{e}_{m'}, \mathbf{e}_m \rangle_{L^2(S^1)}|^2 \right) \sup_{n \in \mathbb{Z}} \left(\sum_{|n'| \leq N} |\langle T_{\mathbf{c}} \mathbf{e}_{n'}, \mathbf{e}_n \rangle_{L^2(S^1)}|^2 \right)} \|G\|_{\text{HS}} \\ &= 2\beta_N^{\mathbf{c}} \|G\|_{\text{HS}}. \end{aligned}$$



For the last equality, we used that $|\langle T_{\mathbf{c}} \mathbf{e}_{n'}, \mathbf{e}_n \rangle_{L^2(S^1)}|^2 = J_{n-n'}^2(k|c|)$ due to the Jacobi–Anger expansion (see, e.g. [13, p 75])

$$e^{\pm i k \hat{\mathbf{x}} \cdot \mathbf{y}} = \sum_{n \in \mathbb{Z}} (\pm i)^n e^{-i n \arg \mathbf{y}} J_n(k|\mathbf{y}|) e^{i n \arg \hat{\mathbf{x}}}, \quad \mathbf{y} \in \mathbb{R}^2, \hat{\mathbf{x}} \in S^1, \quad (2.16)$$

and that $\|(J_m(k|c|))_m\|_{\ell^2} = 1$ (see, e.g. [16, equation (10.23.3)]). Applying the uniform upper bound for the Bessel functions from [34, theorem 2] finally gives the second inequality in (2.15). $\square$

Now we are ready to establish an upper bound for $\|\mathcal{P}_V \mathcal{P}_W\|$ from (2.11).

Proposition 2.4. *For all $G \in \mathcal{V}$ and $H \in \mathcal{W}$ we have that*

$$\frac{|\langle G, H \rangle_{\text{HS}}|}{\|G\|_{\text{HS}} \|H\|_{\text{HS}}} \leq 2 \sqrt{\|\mathcal{T}_{\mathbf{c}_1} G\|_{\ell^0 \times \ell^0} \beta_{N_2}^{c_1-c_2}} \leq \frac{8(2N_1+1)\sqrt{2N_2+1}}{5\sqrt{k}|\mathbf{c}_1 - \mathbf{c}_2|}. \quad (2.17)$$

As outlined at the beginning of this subsection, this bound is useful as long as the right hand side of (2.17) is strictly smaller than one. Since $N_1 \gtrsim kR_1$ and $N_2 \gtrsim kR_2$ are determined by the diameters of the scatterers in wave lengths, the distance $k|\mathbf{c}_1 - \mathbf{c}_2|$ between the two scatterers in wave lengths has to be rather large compared to these diameters for this to be satisfied.

Proof of proposition 2.4. We use lemma 2.3, Hölder’s inequality and the inequality $\|\cdot\| \leq \|\cdot\|_{\text{HS}}$ to estimate

$$\begin{aligned} |\langle G, H \rangle_{\text{HS}}| &= |\langle \mathcal{T}_{\mathbf{c}_1} G, \mathcal{T}_{\mathbf{c}_1 - \mathbf{c}_2}(\mathcal{T}_{\mathbf{c}_2} H) \rangle_{\ell^2 \times \ell^2}| \leq \|\mathcal{T}_{\mathbf{c}_1} G\|_{\ell^1 \times \ell^1} \|\mathcal{T}_{\mathbf{c}_1 - \mathbf{c}_2}(\mathcal{T}_{\mathbf{c}_2} H)\|_{\ell^\infty \times \ell^\infty} \\ &\leq 2 \sqrt{\|\mathcal{T}_{\mathbf{c}_1} G\|_{\ell^0 \times \ell^0} \beta_{N_2}^{c_1-c_2}} \|\mathcal{T}_{\mathbf{c}_1} G\|_{\text{HS}} \|\mathcal{T}_{\mathbf{c}_2} H\|_{\text{HS}} = 2 \sqrt{\|\mathcal{T}_{\mathbf{c}_1} G\|_{\ell^0 \times \ell^0} \beta_{N_2}^{c_1-c_2}} \|G\|_{\text{HS}} \|H\|_{\text{HS}}. \end{aligned}$$

Recalling that $\|\mathcal{T}_{\mathbf{c}_1} G\|_{\ell^0 \times \ell^0} \leq (2N_1+1)^2$ for any $G \in \mathcal{V}$, and using the upper bound for $\beta_{N_2}^{c_1-c_2}$ from (2.15) completes the proof. $\square$

Remark 2.5. Using the Jacobi–Anger expansion (2.16) we can rewrite

$$\beta_{N_2}^{c_1-c_2} = \sqrt{\sup_{n \in \mathbb{Z}} \left(\sum_{|n'| \leq N_2} |\langle T_{-\mathbf{c}_2} \mathbf{e}_{n'}, T_{-\mathbf{c}_1} \mathbf{e}_n \rangle_{L^2(S^1)}|^2 \right)}.$$

This shows that $\beta_{N_2}^{c_1-c_2}$ is a measure of incoherence of $\text{span}\{T_{-\mathbf{c}_2} \mathbf{e}_{n'} : |n'| \leq N_2\} \subset L^2(S^1)$ with respect to the orthonormal system $\{T_{-\mathbf{c}_1} \mathbf{e}_n : n \in \mathbb{Z}\}$ in $L^2(S^1)$. If $\beta_{N_2}^{c_1-c_2}$ is small, this means that operators in $\mathcal{W}$ cannot have a sparse representation in $\mathcal{V}$. We note that $\beta_{N_2}^{c_1-c_2}$ corresponds to $\beta(S)$ as introduced in [8, equation (4.6)] for $S = \text{span}\{T_{-\mathbf{c}_2} \mathbf{e}_{n'} : |n'| \leq N_2\}$ and the n -th standard basis vector replaced by $T_{-\mathbf{c}_1} \mathbf{e}_n$, $n \in \mathbb{Z}$. We also refer to [27], where a similar notion of incoherence was considered.

Our numerical results in figure 2 suggest that the upper bound of $\beta_{N_2}^{c_1-c_2}$ in (2.15) is of optimal order but not sharp. To generate these plots, we have used the superlinear decay of $|J_m(t)|$ in $|m|$ for $|m| > |t|$, which ensures that the supremum in the definition of $\beta_{N_2}^{c_1-c_2}$ is already attained for moderate values of $n \in \mathbb{Z}$, to calculate $\beta_{N_2}^{c_1-c_2}$ numerically. Nevertheless, the upper bound in (2.17) gives a good qualitative impression of sufficient conditions for the geometry of the scatterer to allow for stable splitting. The smaller the diameters of the two components are and the farther apart they are from each other, both in terms of the wave length, the smaller this bound becomes. Here, the size R_1 of the first scatterer has a stronger influence than R_2 . $\diamond$

In the next section we examine a convex program called PCP to approximate F_{q_1} based on *a priori* knowledge of c_1 .

3. Far field operator splitting by PCP

Suppose that (F_1, L) is an approximate split of the far field operator F_q as in (2.10) such that $F_1 \in \mathcal{V}$ is sparse and $L \in \mathcal{W}$ is low-rank. Following [45] we consider the constrained optimization problem

$$\underset{\tilde{F}_1, \tilde{L} \in \text{HS}(L^2(S^1))}{\text{minimize}} \quad \lambda \|\mathcal{T}_{c_1} \tilde{F}_1\|_{\ell^1 \times \ell^1} + \|\tilde{L}\|_{\text{nuc}} \quad \text{subject to} \quad \|F_q^\delta - \tilde{F}_1 - \tilde{L}\|_{\text{HS}} \leq \delta, \quad (3.1)$$

to recover an approximation of (F_1, L) from possibly noisy observations $F_q^\delta \in \text{HS}(L^2(S^1))$ of F_q . Here $\lambda \in (0, 1)$ is a coupling parameter that provides a trade-off between sparse and low-rank components. Its choice will be discussed below. The parameter $\delta > 0$ not just accounts for the data error $F_q^\delta - F_q$ but also for the modeling error $F_q - F_1 - L$ that is caused by the second order Born approximation (2.5) and by the sparse plus low-rank approximation according to (2.10).

In the main result of this work in theorem 3.5 below, we establish the stable recovery of (F_1, L) by means of (3.1). Assuming that the scatterers are very far apart from each other relative to their diameters, we are able to give a rigorous stability estimate. Before we prove this result, we have to consider the same optimization problem but with equality constraints

$$\underset{\tilde{F}_1, \tilde{L} \in \text{HS}(L^2(S^1))}{\text{minimize}} \quad \lambda \|\mathcal{T}_{c_1} \tilde{F}_1\|_{\ell^1 \times \ell^1} + \|\tilde{L}\|_{\text{nuc}} \quad \text{subject to} \quad F_1 + L = \tilde{F}_1 + \tilde{L}, \quad (3.2)$$

which does not take into account modeling and data errors. The latter is commonly referred to as PCP. In proposition 3.1 we give an abstract condition for the unique solvability of (3.2). This optimality condition is stated in terms of a dual vector, the existence of which then certifies optimality. Accordingly, this dual vector is usually called a dual certificate. Certification is a common approach for proving optimality in matrix recovery problems. In proposition 3.3 we then establish conditions on the diameters of the supports of the scatterers D_1 and D_2 and on their distance to each other in terms of the wave length that guarantee the existence of such a dual certificate. We note that in PCP the aim usually is to recover the low-rank matrix L —the principal component—while the sparse matrix component typically is of less interest. However, in our setting it is the other way round as our main goal is to recover the sparse component F_1 , while L typically is not even of very low rank.

For any $\tilde{F}_1, \tilde{L} \in \text{HS}(L^2(S^1))$ we write

$$\Psi_\lambda(\tilde{F}_1, \tilde{L}) := \lambda \|\mathcal{T}_{c_1} \tilde{F}_1\|_{\ell^1 \times \ell^1} + \|\tilde{L}\|_{\text{nuc}}. \quad (3.3)$$

A characterization of the subdifferential of Ψ_λ in (F_1, L) (see, e.g. [8, p 580]) will be required in the proof of proposition 3.1 below (see (3.6)). Denoting the Fourier coefficients in the series representation of $\mathcal{T}_{c_1} F_1$ as in (2.12) by $(a_{m,n}^{c_1})_{m,n}$ and by $(\sigma_n; u_n, v_n)_n$ a singular system for L , we introduce for this purpose the two operators

$$\Sigma := \sum_{|m| \leq N_1} \sum_{|n| \leq N_1} \frac{\overline{a_{m,n}^{c_1}}}{|a_{m,n}^{c_1}|} T_{-c_1} \mathbf{e}_m \langle \cdot, T_{-c_1} \mathbf{e}_n \rangle_{L^2(S^1)} \quad \text{and} \quad \Lambda g := \sum_{n=1}^{2(2N_2+1)} u_n \langle \cdot, v_n \rangle_{L^2(S^1)}. \quad (3.4)$$

The following proposition gives sufficient conditions for the uniqueness of solutions to (3.2). It is similar to [7, lemma 2.5] and [8, proposition 2]. However, since the translation operator $\mathcal{T}_{c_1}$ in (3.3) requires some subtle changes, we include the proof for the sake of completeness.

Proposition 3.1. *The pair (F_1, L) is the unique minimizer of (3.2) if*

- (a) *there holds $\|\mathcal{P}_\mathcal{V} \mathcal{P}_\mathcal{W}\| < 1/2$ and*
- (b) *there exists a dual certificate $W \in \text{HS}(L^2(S^1))$ satisfying*

$$\mathcal{P}_\mathcal{W} W = 0, \quad \|W\| < \frac{1}{2}, \quad \|\mathcal{P}_\mathcal{V}(\Lambda - \lambda \Sigma + W)\|_{\text{HS}} \leq \frac{\lambda}{4}, \quad \|\mathcal{T}_{c_1} \mathcal{P}_{\mathcal{V}^\perp}(\Lambda + W)\|_{\ell^\infty \times \ell^\infty} < \frac{\lambda}{4}. \quad (3.5)$$

Proof. Assume that the conditions (a) and (b) are satisfied. Since any feasible pair $(\tilde{F}_1, \tilde{L})$ for (3.2) can be written as $(F_1 - G, L + G)$ for some $G \in \text{HS}(L^2(S^1))$, we have to show that

$$\Psi_\lambda(F_1 - G, L + G) > \Psi_\lambda(F_1, L)$$

whenever $G \neq 0$. Let $Z_1 \in \partial \|\mathcal{T}_{c_1} F_1\|_{\ell^1 \times \ell^1}$ and $Z_L \in \partial \|L\|_{\text{nuc}}$ be arbitrary subgradients, i.e.

$$Z_1 = \Sigma + G_1 \quad \text{for some } G_1 \in \mathcal{V}^\perp \text{ with } \|\mathcal{T}_{c_1} G_1\|_{\ell^\infty \times \ell^\infty} \leq 1, \quad (3.6a)$$

$$Z_L = \Lambda + G_L \quad \text{for some } G_L \in \mathcal{W}^\perp \text{ with } \|G_L\| \leq 1, \quad (3.6b)$$

(see, e.g. [8, p 580] and the references therein). Then the subgradient property gives

$$\Psi_\lambda(F_1 - G, L + G) \geq \Psi_\lambda(F_1, L) - \lambda \langle Z_1, G \rangle_{\text{HS}} + \langle Z_L, G \rangle_{\text{HS}}. \quad (3.7)$$

We can choose G_1 and G_L satisfying (3.6) such that

$$\langle G_1, \mathcal{P}_{\mathcal{V}^\perp} G \rangle_{\text{HS}} = -\|\mathcal{T}_{c_1} \mathcal{P}_{\mathcal{V}^\perp} G\|_{\ell^1 \times \ell^1} \quad \text{and} \quad \langle G_L, \mathcal{P}_{\mathcal{W}^\perp} G \rangle_{\text{HS}} = \|\mathcal{P}_{\mathcal{W}^\perp} G\|_{\text{nuc}}.$$

For G_1 this can be achieved by picking

$$G_1 = - \sum_{|m| > N_1} \sum_{|n| > N_1} \frac{\overline{b_{m,n}^{c_1}}}{|b_{m,n}^{c_1}|} T_{-c_1} \mathbf{e}_n \langle \cdot, T_{-c_1} \mathbf{e}_n \rangle_{L^2(S^1)}$$

with $(b_{m,n}^{c_1})_{m,n}$ denoting the Fourier coefficients in the series representation of $\mathcal{T}_{c_1} G$ as in (2.12). The operator G_L can be constructed using the duality of $\|\cdot\|$ and $\|\cdot\|_{\text{nuc}}$ as outlined in the proof of [7, lemma 2.5]. Inserting this into (3.7) we obtain

$$\Psi_\lambda(F_1 - G, L + G) \geq \Psi_\lambda(F_1, L) + \Psi_\lambda(\mathcal{P}_{\mathcal{V}^\perp} G, \mathcal{P}_{\mathcal{W}^\perp} G) - |\langle \lambda \Sigma - \Lambda, G \rangle_{\text{HS}}|. \quad (3.8)$$

Rewriting $\Lambda - \lambda \Sigma = \mathcal{P}_{\mathcal{V}}(\Lambda - \lambda \Sigma + W) + \mathcal{P}_{\mathcal{V}^\perp}(\Lambda + W) - W$ and using Hölder's inequality as well as lemma 2.3 (a)–(b) and the assumption (b) of the proposition, we can bound the third term on the right hand side of (3.8) by

$$\begin{aligned} |\langle \lambda \Sigma - \Lambda, G \rangle_{\text{HS}}| &\leq |\langle \mathcal{P}_{\mathcal{V}}(\Lambda - \lambda \Sigma + W), G \rangle_{\text{HS}}| + |\langle \mathcal{P}_{\mathcal{V}^\perp}(\Lambda + W), G \rangle_{\text{HS}}| + |\langle W, G \rangle_{\text{HS}}| \\ &\leq \frac{\lambda}{4} \|\mathcal{P}_{\mathcal{V}} G\|_{\text{HS}} + \frac{\lambda}{4} \|\mathcal{T}_{c_1} \mathcal{P}_{\mathcal{V}^\perp} G\|_{\ell^1 \times \ell^1} + \frac{1}{2} \|\mathcal{P}_{\mathcal{W}^\perp} G\|_{\text{nuc}}. \end{aligned} \quad (3.9)$$

Using condition (a) of the proposition we conclude that

$$\begin{aligned} \|\mathcal{P}_{\mathcal{V}} G\|_{\text{HS}} &\leq \|\mathcal{P}_{\mathcal{V}} \mathcal{P}_{\mathcal{W}} G\|_{\text{HS}} + \|\mathcal{P}_{\mathcal{V}} \mathcal{P}_{\mathcal{W}^\perp} G\|_{\text{HS}} \leq \frac{1}{2} \|G\|_{\text{HS}} + \|\mathcal{P}_{\mathcal{W}^\perp} G\|_{\text{HS}} \\ &\leq \frac{1}{2} \|\mathcal{P}_{\mathcal{V}} G\|_{\text{HS}} + \frac{1}{2} \|\mathcal{P}_{\mathcal{V}^\perp} G\|_{\text{HS}} + \|\mathcal{P}_{\mathcal{W}^\perp} G\|_{\text{HS}}. \end{aligned}$$

Combining this with $\|\cdot\|_{\text{HS}} \leq \|\cdot\|_{\ell^1 \times \ell^1}$ and $\|\cdot\|_{\text{HS}} \leq \|\cdot\|_{\text{nuc}}$ we obtain that

$$\|\mathcal{P}_{\mathcal{V}} G\|_{\text{HS}} \leq \|\mathcal{T}_{c_1} \mathcal{P}_{\mathcal{V}^\perp} G\|_{\ell^1 \times \ell^1} + 2 \|\mathcal{P}_{\mathcal{W}^\perp} G\|_{\text{nuc}}. \quad (3.10)$$

Inserting (3.10) into (3.9) and the result into (3.8) finally yields

$$\Psi_\lambda(F_1 - G, L + G) \geq \Psi_\lambda(F_1, L) + \frac{1}{2} (\lambda \|\mathcal{T}_{c_1} \mathcal{P}_{\mathcal{V}^\perp} G\|_{\ell^1 \times \ell^1} + (1 - \lambda) \|\mathcal{P}_{\mathcal{W}^\perp} G\|_{\text{nuc}}). \quad (3.11)$$

The second term on the right hand side of (3.11) is nonnegative and vanishes if and only if $\mathcal{P}_{\mathcal{V}^\perp} G = \mathcal{P}_{\mathcal{W}^\perp} G = 0$, i.e. if and only if $G = 0$ as $\mathcal{V} \cap \mathcal{W} = \{0\}$ due to condition (a) of the proposition. This ends the proof. $\square$

In proposition 3.3 below, we establish sufficient conditions on λ , k , N_1 , N_2 and $|c_1 - c_2|$ such that the conditions (a) and (b) of proposition 3.1 are fulfilled. For this purpose we define

$$\mu := \sup_{G \in \mathcal{V}, \|\mathcal{T}_{c_1} G\|_{\ell^\infty \times \ell^\infty} \leq 1} \|G\| \quad \text{and} \quad \xi := \sup_{G \in \mathcal{W}, \|G\| \leq 1} \|\mathcal{T}_{c_1} G\|_{\ell^\infty \times \ell^\infty},$$

which correspond to $\mu(F_1)$ and $\xi(L)$ as introduced in [8, equations (1.1), (1.2)]. The number ξ being small means that for any $G \in \mathcal{W}$ the translated operator $\mathcal{T}_{c_1} G$ is not too sparse with respect to the Fourier basis $(\mathbf{e}_n)_n$. A small value for μ would imply that the ranks of the elements of $\mathcal{V}$ are not too small. However, in our case μ will not be very small as we will see in lemma 3.2 below. Nevertheless, we will show that as long as the product of μ and ξ is sufficiently small, we can use arguments developed in [8] to establish the existence of a dual certificate satisfying (3.5). By proposition 3.3 this then guarantees the uniqueness of the minimizer of (3.2). The following lemma expresses μ and ξ in terms of k , N_1 , N_2 and $|c_1 - c_2|$, in terms of quantities determined by the geometrical configuration of the scattering problem.

Lemma 3.2. *There holds*

$$\mu = 2N_1 + 1 \quad \text{and} \quad \xi \leq 2\beta_{N_2}^{c_1 - c_2}$$

with $\beta_{N_2}^{c_1 - c_2}$ as in (2.15).

Proof. The upper bound for ξ follows immediately from (2.14).

For μ we conclude from lemma 2.3 (b) and Parseval's identity for any $G \in \mathcal{V}$ with $(a_{m,n}^{c_1})_{m,n}$ denoting the coefficients of the series expansion of $\mathcal{T}_{c_1}G$ as in (2.12) that

$$\begin{aligned} \|G\| = \|\mathcal{T}_{c_1}G\| &= \sup_{\|f\|_{L^2(S^1)}=1} \|(\mathcal{T}_{c_1}G)f\|_{L^2(S^1)} = \sup_{\|f\|_{L^2(S^1)}=1} \sqrt{\sum_{|m| \leq N_1} |\langle (\mathcal{T}_{c_1}G)f, \mathbf{e}_m \rangle|_{L^2(S^1)}^2} \\ &= \sup_{\|f\|_{L^2(S^1)}=1} \sqrt{\sum_{|m| \leq N_1} \left| \sum_{|n| \leq N_1} a_{m,n}^{c_1} \langle f, \mathbf{e}_n \rangle \right|_{L^2(S^1)}^2} \leq (2N_1 + 1) \|\mathcal{T}_{c_1}G\|_{\ell^\infty \times \ell^\infty}. \end{aligned}$$

This yields $\mu \leq 2N_1 + 1$ and since the upper bound is attained for $f = \sum_{|m| \leq N_1} T_{-c_1} \mathbf{e}_m / (2N_1 + 1)$, we obtain equality. $\square$

Next we establish sufficient conditions on the size of the two scatterers and on the distance between them such that the assumptions (a) and (b) of proposition 3.1 are satisfied. The result is motivated by [8, theorem 2].

Proposition 3.3. *The conditions (a) and (b) of proposition 3.1 are satisfied for*

$$\lambda \in \left(\frac{8\beta_{N_1}^{c_1 - c_2}}{1 - 8(N_1 + 3)(2N_1 + 1)\beta_{N_1}^{c_1 - c_2}}, \frac{2(1 - 16(2N_1 + 1)^2\beta_{N_1}^{c_1 - c_2})}{15(2N_1 + 1)} \right) \quad (3.12)$$

provided $2(2N_1 + 1)^2\beta_{N_1}^{c_1 - c_2} < 1/8$ and provided N_1 and $\beta_{N_1}^{c_1 - c_2}$ are small enough so that above interval is a subset of $(0, 1)$.

Proof. The condition (a) of proposition 3.1 holds due to (2.11) and proposition 2.4 because our assumptions guarantee that $2(2N_1 + 1)\beta_{N_1}^{c_1 - c_2} < 1$.

For (b) we have to show that there exists a dual certificate $W \in \text{HS}(L^2(S^1))$ satisfying (3.5). To this end we follow a similar strategy as was used in the proof of [8, theorem 2]. Since $\mathcal{V} \cap \mathcal{W} = \{0\}$, we can choose $W = \lambda\Sigma + E_1 + E_2 \in \mathcal{V} \oplus \mathcal{W}$ with $E_1 \in \mathcal{V}$ and $E_2 \in \mathcal{W}$ such that

$$\mathcal{P}_{\mathcal{W}}W = 0 \quad \text{and} \quad \|\mathcal{P}_{\mathcal{V}}(\Lambda - \lambda\Sigma + W)\|_{\text{HS}} \leq \frac{\lambda}{4}.$$

From (3.4) we immediately obtain that

$$\|\mathcal{T}_{c_1}\Sigma\|_{\ell^\infty \times \ell^\infty} = \|\Lambda\| = 1, \quad (3.13)$$

and since $\mathcal{P}_{\mathcal{W}}W = 0$, we conclude that $E_2 = -\mathcal{P}_{\mathcal{W}}(\lambda\Sigma + E_1)$.

Next, we denote by $P_U : L^2(S^1) \rightarrow L^2(S^1)$ the orthogonal projection onto

$$U := \text{span}\{T_{-c_2}\mathbf{e}_n \mid |n| \leq N_2\}.$$

Then, we can rewrite $\mathcal{P}_{\mathcal{W}}G = P_UG + GP_U - P_UGP_U$ for any $G \in \text{HS}(L^2(S^1))$, which implies $\|\mathcal{P}_{\mathcal{W}}G\| \leq \|P_UG\| + \|(I - P_U)GP_U\| \leq 2\|G\|$. Combining this with the definition of μ and (3.13) we obtain that

$$\|E_2\| \leq 2\mu(\lambda + \|\mathcal{T}_{c_1}E_1\|_{\ell^\infty \times \ell^\infty}). \quad (3.14)$$

Furthermore, we deduce from

$$\begin{aligned} \frac{\lambda}{4} &\geq \|\mathcal{P}_{\mathcal{V}}(\Lambda - \lambda\Sigma + W)\|_{\text{HS}} \geq -(2N_1 + 1)\|\mathcal{T}_{c_1}\mathcal{P}_{\mathcal{V}}(\Lambda + E_2)\|_{\ell^\infty \times \ell^\infty} + \|\mathcal{T}_{c_1}E_1\|_{\ell^\infty \times \ell^\infty} \\ &\geq -(2N_1 + 1)\|\mathcal{T}_{c_1}(\Lambda + E_2)\|_{\ell^\infty \times \ell^\infty} + \|\mathcal{T}_{c_1}E_1\|_{\ell^\infty \times \ell^\infty}, \end{aligned}$$

the definition of ξ and (3.13) that

$$\|\mathcal{T}_{c_1}E_1\|_{\ell^\infty \times \ell^\infty} \leq \frac{\lambda}{4} + (2N_1 + 1)(1 + \|E_2\|)\xi. \quad (3.15)$$

Combining lemma 3.2 with the assumptions of the proposition, we obtain that

$$2(2N_1 + 1)\mu\xi < 4(2N_1 + 1)^2\beta_{N_1}^{c_1 - c_2} < 1.$$

Accordingly, substituting (3.14) into (3.15) and solving for $\|\mathcal{T}_{c_1}E_1\|_{\ell^\infty \times \ell^\infty}$ gives

$$\|\mathcal{T}_{c_1}E_1\|_{\ell^\infty \times \ell^\infty} \leq \frac{(2N_1 + 1)\xi + (2(2N_1 + 1)\mu\xi + \frac{1}{4})\lambda}{1 - 2(2N_1 + 1)\mu\xi}. \quad (3.16)$$

Similarly, inserting (3.16) into (3.14) yields

$$\|E_2\| \leq \frac{\frac{5}{2}\mu\lambda + 2(2N_1 + 1)\mu\xi}{1 - 2(2N_1 + 1)\mu\xi}. \quad (3.17)$$

The definition of μ and (3.13) show that

$$\|W\| \leq \mu\lambda\|\mathcal{T}_{c_1}\Sigma\|_{\ell^\infty \times \ell^\infty} + \mu\|\mathcal{T}_{c_1}E_1\|_{\ell^\infty \times \ell^\infty} + \|E_2\| \leq \mu(\lambda + \|\mathcal{T}_{c_1}E_1\|_{\ell^\infty \times \ell^\infty}) + \|E_2\|, \quad (3.18)$$

and using the definition of ξ and again (3.13) gives

$$\|\mathcal{T}_{c_1}\mathcal{P}_{\mathcal{V}^\perp}(\Lambda + W)\|_{\ell^\infty \times \ell^\infty} \leq \|\mathcal{T}_{c_1}(\Lambda + E_2)\|_{\ell^\infty \times \ell^\infty} \leq \xi(1 + \|E_2\|). \quad (3.19)$$

Inserting the upper bounds (3.16) and (3.17) into (3.18) and (3.19), we conclude that

$$\|W\| < \frac{1}{2} \quad \text{and} \quad \|\mathcal{T}_{c_1}\mathcal{P}_{\mathcal{V}^\perp}(\Lambda + W)\|_{\ell^\infty \times \ell^\infty} < \frac{\lambda}{4}$$

if

$$\lambda \in \left(\frac{4\xi}{1 - 4(N_1 + 3)\mu\xi}, \frac{2(1 - 16(2N_1 + 1)\mu\xi)}{15\mu} \right),$$

i.e. we have constructed a dual certificate $W \in \text{HS}(L^2(S^1))$ satisfying (3.5). Applying lemma 3.2 once more finally gives (3.12). $\square$

Remark 3.4. Unfortunately, already for rather small values of $N_1 \gtrsim kR_1$, the value of $k|c_1 - c_2|$ must be extremely large for the assumptions of proposition 3.3 to be fulfilled. For instance, for $k = 1$, $R_1 = R_2 = 5$ and $N_1 = N_2 = 7$ as in example 2.1, the condition $2(2N_1 + 1)^2\beta_{N_1}^{c_1 - c_2} < 1/8$ would require $|c_1 - c_2|$ of the order of 4×10^6 . However, our numerical reconstructions below illustrate that even when the assumptions of proposition 3.3 are not met, far field operator splitting by PCP still gives satisfactory results. This suggests that proposition 3.3 is far from being sharp. $\diamond$

Now we turn to the relaxed minimization problem (3.1), which takes into account data errors and modeling errors, and show a stability estimate for far field operator splitting with noisy data. A related stability estimate for sparse plus low-rank matrix splitting has been established in [45, proposition 4], and we adapt and generalize this result for the far field operator splitting problem considered here. In theorem 3.5 below $F_q^0 \in \mathcal{V}_N^{c_1}$ represents a discrete version of the far field operator $F_q \in \text{HS}(L^2(S^1))$. Since we assume throughout this work that the whole ensemble of scatterers is compactly supported, there exists a smallest radius $R > 0$ such that $D \subseteq B_R(c_1)$ is contained inside the ball of radius R around c_1 . Accordingly, the results from [22] say that choosing $N \gtrsim kR$ is sufficient to obtain an accurate best approximation of F_q in $\mathcal{V}_N^{c_1}$. Furthermore, $F_q^\delta \in \mathcal{V}_N^{c_1}$ denotes a discretized noisy observation of F_q . The bound δ_0 in (3.20) accounts for the approximation error of the discretization of F_q by F_q^0 and also for the modeling error $F_q - F_1 - L$ of our sparse plus low-rank ansatz from (2.10). On the other hand, the data error $F_q^0 - F_q^\delta$ is bounded by (3.21), which also guarantees (F_1, L) to be feasible for problem (3.22). We note that the only *a priori* information required by (3.22) is the approximate position c_1 of the component D_1 of the scatterer.

Theorem 3.5. Suppose that $F_q^0, F_q^\delta \in \mathcal{V}_N^{c_1}$ for some $N \in \mathbb{N}$, and let $c_1, c_2 \in \mathbb{R}^2$ and $N_1, N_2 \in \mathbb{N}$ with $N_1, N_2 \leq N$. Let $C_{N_1}^{c_1 - c_2} := 2(2N_1 + 1)\beta_{N_1}^{c_1 - c_2} < 1/2$ and assume that there exists a dual certificate satisfying the conditions in proposition 3.1. Furthermore, we assume that $F_1 \in \mathcal{V}$ and $L \in \mathcal{W}$ are such that

$$\max\{\|F_q^0 - (F_1 + L)\|_{\ell^1 \times \ell^1}, \|F_q^0 - (F_1 + L)\|_{\text{nuc}}, \|(\mathcal{I} - \mathcal{P}_{\mathcal{V}_N^{c_1}})L\|_{\text{nuc}}\} \leq \delta_0 \quad (3.20)$$

for some $\delta_0 \geq 0$. Suppose that $\delta > 0$ satisfies

$$\delta \geq 2\delta_0 + \|F_q^0 - F_q^\delta\|_{\text{HS}} \quad (3.21)$$

and let $(F_1^\delta, L^\delta) \in \mathcal{V}_{N_1}^{c_1} \times \mathcal{V}_N^{c_1}$ denote a solution to

$$\text{minimize}_{\tilde{F}_1, \tilde{L} \in \mathcal{V}_N^{c_1}} \Psi_\lambda(\tilde{F}_1, \tilde{L}) \quad \text{subject to} \quad \|F_q^\delta - (\tilde{F}_1 + \tilde{L})\|_{\text{HS}} \leq \delta. \quad (3.22)$$

Here, we choose $\lambda = \eta/\sqrt{2N+1} \leq 1/2$ for some $\eta \geq 1$. Then, there exists a constant $C > 0$ such that

$$\|F_1 - F_1^\delta\|_{\text{HS}}^2 \leq C \left(1 + (1 + (1 - C_{N_1}^{c_1 - c_2})^{-1})(2N+1)^2\right) \delta^2 \quad (3.23a)$$

and

$$\|L - L^\delta\|_{\text{HS}}^2 \leq C \left(1 + (1 + (1 - C_{N_1}^{c_1 - c_2})^{-1})(2N+1)^2\right) \delta^2. \quad (3.23b)$$

Proof. We use the notations $\hat{F}_1 := F_1 - F_1^\delta$, $\hat{L} := L - L^\delta$, $\hat{G}^+ := (\hat{F}_1 + \hat{L})/2$ and $\hat{G}^- := (\hat{F}_1 - \hat{L})/2$. From the parallelogram identity we conclude

$$\begin{aligned} \|\hat{F}_1\|_{\text{HS}}^2, \|\hat{L}\|_{\text{HS}}^2 &\leq \|\hat{F}_1\|_{\text{HS}}^2 + \|\hat{L}\|_{\text{HS}}^2 = 2\|\hat{G}^+\|_{\text{HS}}^2 + 2\|\hat{G}^-\|_{\text{HS}}^2 \\ &= 2\|\hat{G}^+\|_{\text{HS}}^2 + (\|\mathcal{P}_V \hat{G}^-\|_{\text{HS}}^2 + \|\mathcal{P}_W \hat{G}^-\|_{\text{HS}}^2) + (\|\mathcal{P}_{V^\perp} \hat{G}^-\|_{\text{HS}}^2 + \|\mathcal{P}_{W^\perp} \hat{G}^-\|_{\text{HS}}^2). \end{aligned} \quad (3.24)$$

In the following, we bound the three terms on the right hand side of (3.24) separately.

Applying (3.20), (3.21) and the fact that $\|\cdot\|_{\text{HS}} \leq \|\cdot\|_{\text{nuc}}$ we obtain for the first term on the right hand side of (3.24) that

$$\begin{aligned} 2\|\hat{G}^+\|_{\text{HS}}^2 &\leq \frac{1}{2} (\|F_q^0 - (F_1 + L)\|_{\text{HS}} + \|F_q^0 - F_q^\delta\|_{\text{HS}} + \|F_q^\delta - (F_1^\delta + L^\delta)\|_{\text{HS}})^2 \\ &\leq \frac{1}{2} (2\delta - \delta_0)^2 \leq 2\delta^2. \end{aligned} \quad (3.25)$$

To bound the second term on the right hand side of (3.24) we note first that, using (2.17),

$$\begin{aligned} \|\mathcal{P}_V \hat{G}^- - \mathcal{P}_W \hat{G}^-\|_{\text{HS}}^2 &\geq \|\mathcal{P}_V \hat{G}^-\|_{\text{HS}}^2 + \|\mathcal{P}_W \hat{G}^-\|_{\text{HS}}^2 - 2|\langle \mathcal{P}_V \hat{G}^-, \mathcal{P}_W \hat{G}^- \rangle|_{\text{HS}} \\ &\geq \|\mathcal{P}_V \hat{G}^-\|_{\text{HS}}^2 + \|\mathcal{P}_W \hat{G}^-\|_{\text{HS}}^2 - 2C_{N_1}^{c_1 - c_2} \|\mathcal{P}_V \hat{G}^-\|_{\text{HS}} \|\mathcal{P}_W \hat{G}^-\|_{\text{HS}} \\ &\geq (1 - C_{N_1}^{c_1 - c_2}) (\|\mathcal{P}_V \hat{G}^-\|_{\text{HS}}^2 + \|\mathcal{P}_W \hat{G}^-\|_{\text{HS}}^2). \end{aligned}$$

Dividing this inequality by $1 - C_{N_1}^{c_1 - c_2}$, which is valid since we assumed that $C_{N_1}^{c_1 - c_2} < 1/2$, and estimating further gives

$$\begin{aligned} \|\mathcal{P}_V \hat{G}^-\|_{\text{HS}}^2 + \|\mathcal{P}_W \hat{G}^-\|_{\text{HS}}^2 &\leq \frac{1}{1 - C_{N_1}^{c_1 - c_2}} \|\mathcal{P}_V \hat{G}^- - \mathcal{P}_W \hat{G}^-\|_{\text{HS}}^2 \\ &= \frac{1}{1 - C_{N_1}^{c_1 - c_2}} \|\mathcal{P}_{V^\perp} \hat{G}^- - \mathcal{P}_{W^\perp} \hat{G}^-\|_{\text{HS}}^2 \\ &\leq \frac{2}{1 - C_{N_1}^{c_1 - c_2}} (\|\mathcal{P}_{V^\perp} \hat{G}^-\|_{\text{HS}}^2 + \|\mathcal{P}_{W^\perp} \hat{G}^-\|_{\text{HS}}^2), \end{aligned} \quad (3.26)$$

i.e. it remains to establish a bound for the third term on the right hand side of (3.24).

Using the isometry property of $\mathcal{T}_{c_1}$ on $\text{HS}(L^2(S^1))$ and the estimates $\|\cdot\|_{\text{HS}} \leq \|\cdot\|_{\text{nuc}}$ and $\|\cdot\|_{\text{HS}} \leq \|\cdot\|_{\ell^1 \times \ell^1}$ it follows for $\lambda \geq 1/\sqrt{2N+1}$ that

$$\begin{aligned} \|\mathcal{P}_{V^\perp} \hat{G}^-\|_{\text{HS}}^2 + \|\mathcal{P}_{W^\perp} \hat{G}^-\|_{\text{HS}}^2 &\leq (\|\mathcal{P}_{V^\perp} \hat{G}^-\|_{\text{HS}} + \|\mathcal{P}_{W^\perp} \hat{G}^-\|_{\text{HS}})^2 \\ &\leq (2N+1) (\Psi_\lambda(\mathcal{P}_{V^\perp} \hat{G}^-, \mathcal{P}_{W^\perp} \hat{G}^-))^2. \end{aligned} \quad (3.27)$$

Since $(F_1, \mathcal{P}_{V_N^{c_1}} L)$ is feasible for (3.22) by (3.20), we conclude using (3.20) once more that

$$\Psi_\lambda(F_1 - \hat{F}_1, L - \hat{L}) = \Psi_\lambda(F_1^\delta, L^\delta) \leq \Psi_\lambda(F_1, \mathcal{P}_{V_N^{c_1}} L) \leq \Psi_\lambda(F_1, L) + \delta_0. \quad (3.28)$$

Next we use (3.11) with $G = \hat{G}^-$, which is valid since we assumed the assumptions of proposition 3.1 to be satisfied, and recall that $\lambda \leq 1/2$ to conclude that

$$\begin{aligned} \Psi_\lambda(F_1 - \hat{G}^-, L + \hat{G}^-) - \psi_\lambda(F_1, L) &\geq \frac{1}{2} \left(\lambda \|\mathcal{T}_{c_1} \mathcal{P}_{V^\perp} \hat{G}^-\|_{\ell^1 \times \ell^1} + \frac{1}{2} \|\mathcal{P}_{W^\perp} \hat{G}^-\|_{\text{nuc}} \right) \\ &\geq \frac{1}{4} \Psi_\lambda(\mathcal{P}_{V^\perp} \hat{G}^-, \mathcal{P}_{W^\perp} \hat{G}^-). \end{aligned}$$

Consequently, we obtain by rewriting $\widehat{G}^- = \widehat{F}_1 - \widehat{G}^+ = -\widehat{L} + \widehat{G}^+$ that

$$\begin{aligned}\Psi_\lambda(\mathcal{P}_{\mathcal{V}^\perp} \widehat{G}^-, \mathcal{P}_{\mathcal{W}^\perp} \widehat{G}^-) &\leq 4(\Psi_\lambda(F_1 - \widehat{G}^-, L + \widehat{G}^-) - \psi_\lambda(F_1, L)) \\ &\leq 4(\Psi_\lambda(F_1 - \widehat{F}_1, L - \widehat{L}) - (\Psi_\lambda(F_1, L) + \delta_0) + (\Psi_\lambda(\widehat{G}^+, \widehat{G}^+) + \delta_0)) \\ &\leq 4(\Psi_\lambda(\widehat{G}^+, \widehat{G}^+) + \delta_0),\end{aligned}\quad (3.29)$$

where we have used (3.28) in the last step. Now we insert (3.29) into (3.27) to deduce

$$\begin{aligned}\|\mathcal{P}_{\mathcal{V}^\perp} \widehat{G}^-\|_{\text{HS}}^2 + \|\mathcal{P}_{\mathcal{W}^\perp} \widehat{G}^-\|_{\text{HS}}^2 &\leq 16(2N+1)(\Psi_\lambda(\widehat{G}^+, \widehat{G}^+) + \delta_0)^2 \\ &\leq 4(2N+1)(\Psi_\lambda(F_q^0 - (F_1^\delta + L^\delta), F_q^0 - (F_1^\delta + L^\delta)) \\ &\quad + \Psi_\lambda(F_q^0 - (F_1 + L), F_q^0 - (F_1 + L)) + 2\delta_0)^2.\end{aligned}\quad (3.30)$$

Since $\|\cdot\|_{\text{nuc}} \leq \sqrt{2N+1}\|\cdot\|_{\text{HS}}$ and $\|\mathcal{T}_{c_1}(\cdot)\|_{\ell^1 \times \ell^1} \leq (2N+1)\|\cdot\|_{\text{HS}}$ on $\mathcal{V}_N^{c_1}$, we can use (3.20) and (3.21) to obtain

$$\Psi_\lambda(F_q^0 - (F_1^\delta + L^\delta), F_q^0 - (F_1^\delta + L^\delta)) \leq 2(\sqrt{2N+1} + \lambda(2N+1))(\delta - \delta_0).$$

Similarly,

$$\Psi_\lambda(F_q^0 - (F_1 + L), F_q^0 - (F_1 + L)) \leq (1 + \lambda)\delta_0.$$

Substituting the last two inequalities into (3.30) gives

$$\begin{aligned}\|\mathcal{P}_{\mathcal{V}^\perp} \widehat{G}^-\|_{\text{HS}}^2 + \|\mathcal{P}_{\mathcal{W}^\perp} \widehat{G}^-\|_{\text{HS}}^2 &\leq 4(2N+1)(2(\sqrt{2N+1} + \lambda(2N+1))\delta - (2\sqrt{2N+1} + \lambda(4N+1) - 3)\delta_0)^2 \\ &\leq 16(2N+1)(\sqrt{2N+1} + \lambda(2N+1))^2 \delta^2.\end{aligned}\quad (3.31)$$

Finally, we insert (3.25), (3.26) and (3.31) into (3.24) to obtain

$$\|F_1 - F_1^\delta\|_{\text{HS}}^2, \|L - L^\delta\|_{\text{HS}}^2 \leq (2 + (1 + 2(1 - C_{N_1}^{c_1 - c_2})^{-1})16(2N+1)(\sqrt{2N+1} + \lambda(2N+1))^2) \delta^2,$$

which implies (3.23) by our choice of λ . $\square$

4. Numerical examples

Now we briefly comment on the numerical implementation of far field operator splitting by PCP as discussed in the previous section. Following [44] we consider an unconstrained version of (3.1), where the inequality constraint is replaced by a penalty term. Accordingly, we aim to solve

$$\text{minimize}_{\widetilde{F}_1, \widetilde{L} \in \mathcal{V}_N^{c_1}} \|F_q^\delta - (\widetilde{F}_1 + \widetilde{L})\|_{\text{HS}}^2 + \tau(\lambda \|\mathcal{T}_{c_1} \widetilde{F}_1\|_{\ell^1 \times \ell^1} + \|\widetilde{L}\|_{\text{nuc}}), \quad (4.1)$$

where $\tau > 0$ is constant. It is well-known that (4.1) is equivalent to (3.1) for some value $\tau(\delta) > 0$, and that $\tau(\delta)$ should be chosen larger for increasing noise levels δ . In the numerical examples below we choose the value of $\tau(\delta)$ by trial and error. The coupling parameter λ controls the balance between promoting sparsity of $\widetilde{F}_1$ and the low-rank structure of $\widetilde{L}$. We use a proximal gradient approach as proposed in [35, 44] to approximate the unique solution of (4.1). Algorithm 1 describes this method in more detail. Here, $\mathcal{D}_{\omega\tau}$ denotes soft singular value thresholding at threshold level $\omega\tau/2$ and $\mathcal{M}_{\omega\tau\lambda, k, c_1}$ elementwise soft thresholding of the modulated Fourier coefficients at threshold level $\omega\tau\lambda/2$. As a stopping criterion we choose the one proposed in [35], i.e. we terminate the iteration if $\|S_{F_1}^{(l+1)}\|_{\text{HS}}^2 + \|S_L^{(l+1)}\|_{\text{HS}}^2$ falls below a prescribed tolerance, where

$$\begin{aligned}S_{F_1}^{(l+1)} &= \frac{1}{\omega}(H_{F_1}^{(l)} - F_1^{(l+1)}) + (F_1^{(l+1)} + L^{(l+1)} - H_{F_1}^{(l)} - H_L^{(l)}), \\ S_L^{(l+1)} &= \frac{1}{\omega}(H_L^{(l)} - L^{(l+1)}) + (F_1^{(l+1)} + L^{(l+1)} - H_{F_1}^{(l)} - H_L^{(l)}).\end{aligned}$$

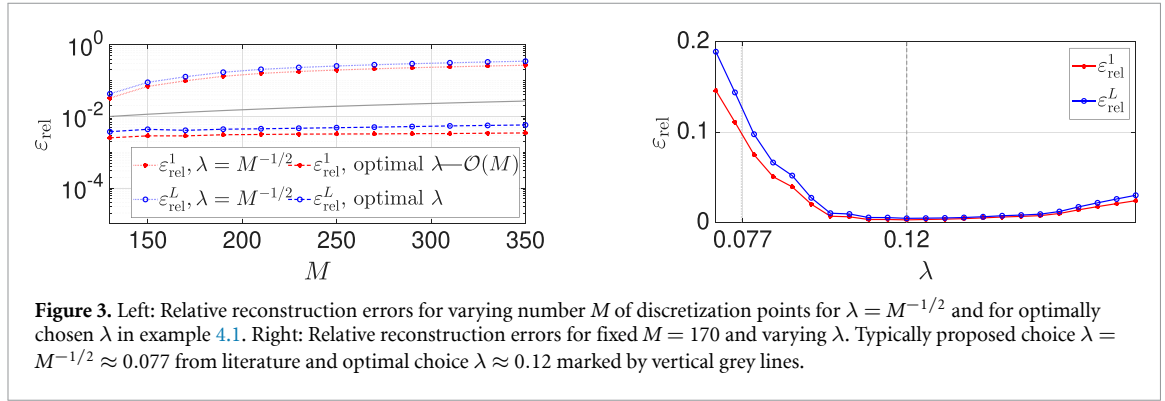
In all our numerical examples apart from the final one the contrast function is chosen to be $q = \chi_{D_1} - 0.5\chi_{D_2}$ with two scatterers $D_1 \subset B_{R_1}(c_1)$ and $D_2 \subset B_{R_2}(c_2)$ as shown in figure 1 (left), and we use $k = 1$ for the wave number. We simulate the associated far field operators F_q and F_{q_1} using a Nyström method with $M = 170$ equidistant illumination and observation directions on S^1 as described

Algorithm 1. Far field operator splitting by principal component pursuit.

```

1: function SPLITTING ( $F_q^\delta, k, c_1, \tau, \lambda$ )
2:   Set  $\omega = 1/3, t_0 = 1, F_1^{(0)} = L^{(0)} = H_{F_1}^{(0)} = H_L^{(0)} = \mathbf{0}, R^{(0)} = F_q^\delta$ .
3:   for  $l = 0, \dots$  do
4:     Compute
5:      $F_1^{(l+1)} = \mathcal{D}_{\omega\tau}(H_L^{(l)} + \omega R^{(l)}), L^{(l+1)} = \mathcal{M}_{\omega\tau\lambda, k c_1}(H_{F_1}^{(l)} + \omega R^{(l)}),$ 
6:      $t_{l+1} = \frac{1 + \sqrt{1 + 4t_l^2}}{2},$ 
7:      $H_{F_1}^{(l+1)} = F_1^{(l)} + \frac{t_l - 1}{t_{l+1}}(F_1^{(l+1)} - F_1^{(l)}), H_L^{(l+1)} = L^{(l)} + \frac{t_l - 1}{t_{l+1}}(L^{(l+1)} - L^{(l)}),$ 
8:      $R^{(l+1)} = F_q^\delta - (H_{F_1}^{(l+1)} + H_L^{(l+1)}).$ 
9:   Terminate on convergence or when  $l$  attains a prescribed  $l_{\max}$ .
10:  end for
11: end function

```



in example 2.1. Here, the number of sampling points M has been chosen such that all far field operators in the examples below can be fully resolved.

Denoting in the following the numerical solutions of (4.1) by $(\hat{F}_1, \hat{L})$, we will evaluate the relative reconstruction errors

$$\varepsilon_{\text{rel}}^1 := \frac{\|F_{q_1} - \hat{F}_1\|_{\text{HS}}}{\|F_{q_1}\|_{\text{HS}}} \quad \text{and} \quad \varepsilon_{\text{rel}}^L := \frac{\|(F_q - F_{q_1}) - \hat{L}\|_{\text{HS}}}{\|F_q - F_{q_1}\|_{\text{HS}}}$$

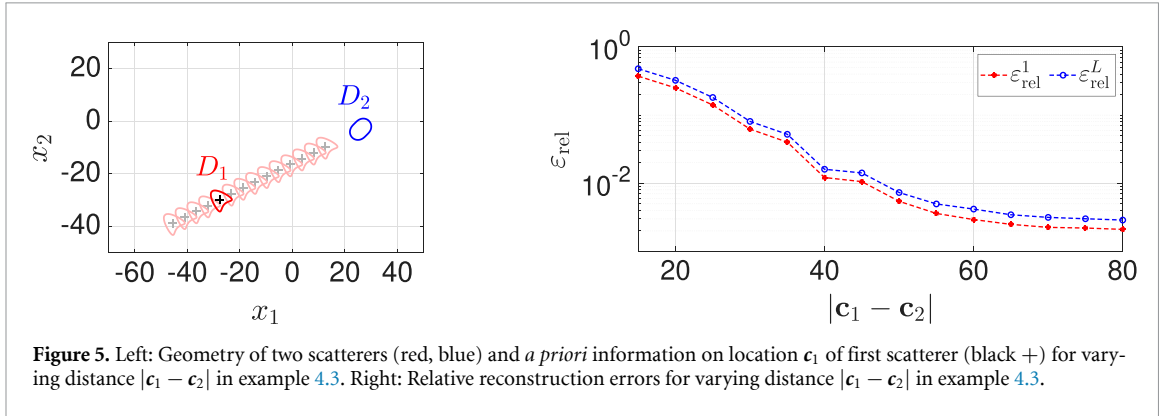
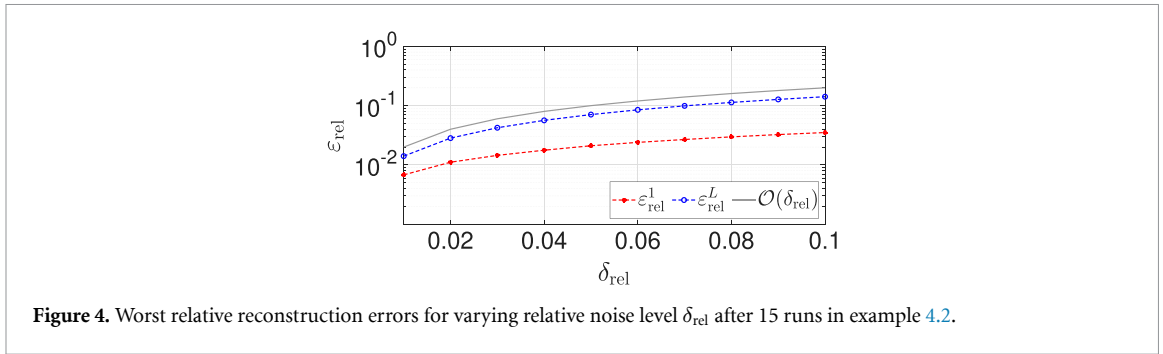
to assess the quality of these numerical solutions. The *a priori* information $c_1 = (-28, -30)^\top$ on the approximate location of the scatterer D_1 that enters the reconstruction algorithm is marked by a cross inside D_1 in figure 1 (left).

We note that the assumptions of theorem 3.5 are not satisfied for the particular setup discussed in this section because the two scatterers are too close to each other relative to their diameters. However, we will see that even for this challenging setup the splitting algorithm still gives good reconstructions.

Example 4.1 (Choice of coupling parameter λ). In our first example, we study the optimal choice of the coupling parameter λ in (4.1) depending on the number of incident and observation directions M used to approximate the far field operators. To this end we vary M between $M = 130$ and $M = 350$, always using $\tau = 3.5 \times 10^{-5} M / \lambda$ for the second parameter in (4.1).

In the literature typically the value $\lambda = M^{-1/2}$ is proposed (see, e.g. [7, 45]). This is also the smallest possible choice of λ that is covered by our stability theorem 3.5. However, since a larger choice of λ promotes a better reconstruction quality for the sparse component F_1 , it makes sense to choose λ larger. We have determined the corresponding optimal values for λ depending on M by minimizing the associated relative reconstruction error $\varepsilon_{\text{rel}}^1$ of the sparse component. The results are shown in figure 3 (left). It turns out that choosing $\lambda \approx 2M^{-1/2}$ always produces better reconstructions. For all error curves in figure 3 (left) we observe an increase of the order $\mathcal{O}(M)$ with respect to M , as to be expected from (3.23).

Figure 3 (right) shows the relative reconstruction errors depending on the coupling parameter λ for fixed number of incident and observation directions $M = 170$. The literature value $\lambda = M^{-1/2} \approx 0.0767$ as well as our optimized choice $\lambda \approx 0.12$ are marked by vertical grey lines. ◇

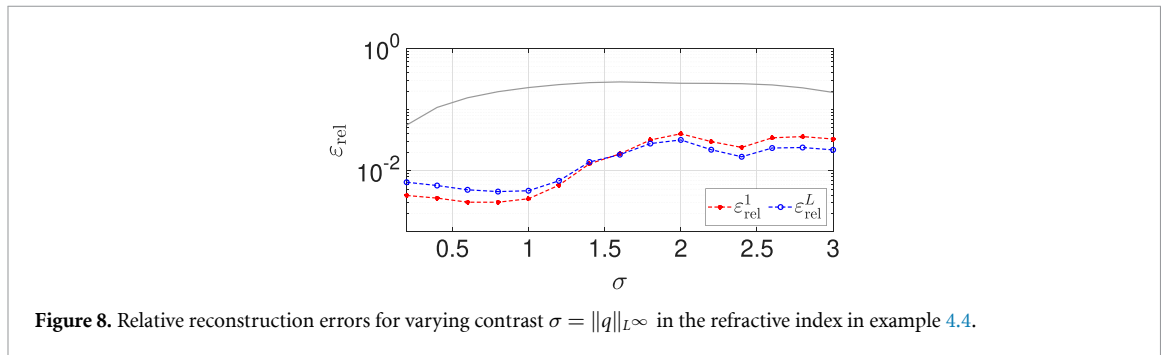
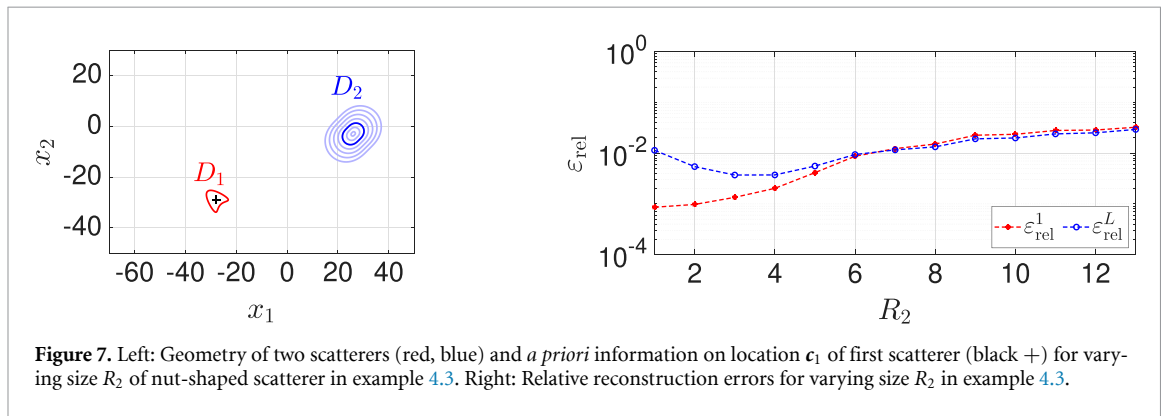
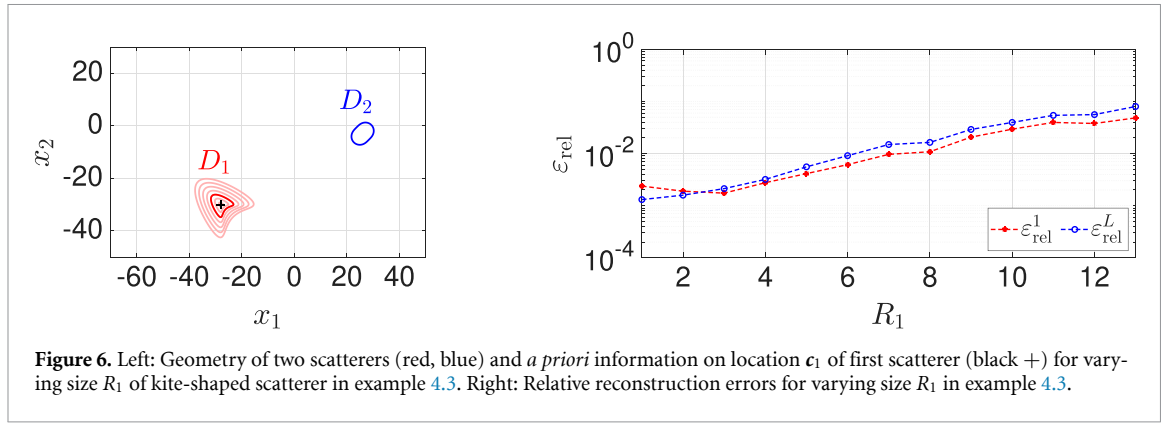


Example 4.2 (Varying noise level δ). We consider the same setting as in the previous example with fixed $M = 170$ and choose accordingly $\lambda = 0.12$. In this example we study the quality of our reconstructions when complex valued uniformly distributed relative error is being added to the simulated far field operators F_q . Denoting by $\delta_{\text{rel}} \in (0, 0.1)$ the relative noise level, we generate 15 different random noise realizations for each noise level and plot the relative reconstruction errors of the worst reconstructions in figure 4. We note that larger noise levels require more regularization, and accordingly the parameter τ in (4.1) has been increased from $\tau = 3.5 \times 10^{-5}M/\lambda$ for $\delta_{\text{rel}} = 0$ to $\tau = 1.25 \times 10^{-4}M/\lambda$ for $\delta_{\text{rel}} = 0.1$ in these calculations. As soon as the additive random noise dominates the modeling error, both error curves in figure 4 increase with a rate of order $\mathcal{O}(\delta_{\text{rel}})$ with respect to δ_{rel} , as expected from (3.23). $\diamond$

Example 4.3 (Varying distance between and diameter of scatterers). In our third example, we study the dependence of the performance of our method depending on the distance $|\mathbf{c}_1 - \mathbf{c}_2|$ between the two components of the scatterer and on the size parameters R_1 and R_2 . In our stability estimates (3.23) these dependencies are hidden in the constant $C_{N_1}^{\mathbf{c}_1 - \mathbf{c}_2}$ (see also the last estimate at the end of the proof of theorem 3.5). Accordingly, the accuracy of the numerical reconstructions should improve with increasing distance and with decreasing radii. Our estimates also suggest that the relative reconstruction errors should grow faster for increasing R_1 than for increasing R_2 . In our numerical tests below, we use $M = 170$ and choose $\tau = 3.5 \times 10^{-5}M/\lambda$.

We start by varying the distance $|\mathbf{c}_1 - \mathbf{c}_2|$ between the scatterers. To this end we keep the position $\mathbf{c}_2$ of the nut-shaped scatterer D_2 fixed and vary the position $\mathbf{c}_1$ of the kite-shaped scatterer D_1 , as shown in figure 5 (left). The instance for $|\mathbf{c}_1 - \mathbf{c}_2| = 60$ is highlighted. The resulting relative reconstruction errors with optimally chosen coupling parameter λ (see example 4.1) are shown in figure 5 (right). As expected, both relative errors decrease with increasing distance $|\mathbf{c}_1 - \mathbf{c}_2|$. In theorem 3.5, the condition $C_{N_1}^{\mathbf{c}_1 - \mathbf{c}_2} < 1/2$ is satisfied for $R_1 = R_2 = 5$, so $N_1 = N_2 = 7$, only for distances $|\mathbf{c}_1 - \mathbf{c}_2|$ on the order of 7000.

In order to analyze the quality of our reconstructions depending on the size of the scatterers, we fix one of the two and vary the size of the other. In both tests, we select the coupling parameter λ optimally (see example 4.1). We start by varying the size R_1 of the kite-shaped scatterer as shown in figure 6 (left), where its instance for $R_1 = 5$ is highlighted. The associated relative reconstruction errors are plotted in figure 6 (right). For $R_1 \geq 3$ both error curves increase with increasing R_1 , as expected. For $R_1 < 3$, the relative error $\varepsilon_{\text{rel}}^1$ increases as R_1 decreases. At $R_1 = 1$, the low-rank component L is reconstructed even better than the sparse component F_1 . This behavior is probably due to the fact that for R_1 very small q_2 scatters much stronger than q_1 . Finally, we vary the size R_2 of the nut-shaped scatterer as shown in figure 7 (left). Again the instance corresponding to $R_2 = 5$ is highlighted. The associated relative reconstruction errors can



be found in figure 6 (right). As expected, we observe $\varepsilon_{\text{rel}}^1$ to grow with R_2 . It can also be seen, that increasing R_1 has a stronger effect on the relative reconstruction error than increasing R_2 , in agreement with our theoretical results. $\diamond$

Example 4.4 (Varying intensity of contrast). As a final example, we study how our method performs as multiple scattering effects constitute an increasingly large portion of the observed far field data. We consider $q = \sigma(\chi_{D_1} + 0.5\chi_{D_2})$ for $\sigma = \|q\|_{L^\infty} \in [0.2, 3]$ for the same scatterer geometry, wave number $k = 1$ and resolution $M = 170$ as before. We use $\lambda = 0.125$ for the coupling parameter and choose the regularization parameter $\tau > 0$ by trial and error with values between $\tau = 10^{-5}M/\lambda$ and $\tau = 4.5 \times 10^{-5}M/\lambda$. Although the modeling error itself is not directly accessible, the relative quantity $\|F_q - (F_{q_1} - F_{q_2})\|_{\text{HS}}/\|F_q\|_{\text{HS}}$ provides an approximate indication of its magnitude. It quantifies the relative contribution of the multiple scattering components in F_q included neither in F_{q_1} nor F_{q_2} . It is shown in gray together with the resulting relative errors in figure 8. Above quantity increases initially in σ until it reaches a plateau for $\sigma \geq 1.6$. A similar trend is reflected in the error curves. The latter show a very good reconstruction quality for $\sigma \leq 1.2$ and increasing reconstruction errors thereafter, which eventually reach a plateau for $\sigma \geq 2$. $\diamond$

5. Conclusions

We have developed a new method to separate or split off the scattering data associated to a single scatterer that is part of an ensemble of well-separated scatterers from the scattering data for the whole ensemble. This question arises for instance when one wishes to recover properties of scatterers in a certain focus area while not being interested in scatterers outside this region so much. Using sparsity and low-rank properties of far field operators associated to compactly supported scatterers, we have shown that a convex program called principle component pursuit can be utilized to approximate solutions to this inverse problem. Our main theoretical result is a stability estimate for this method taking into account modeling errors and data noise. We have put particular emphasis on expressing the prerequisites of this theorem and the associated stability constants solely in terms of geometric properties of the scattering ensemble and the wave length. Although the assumptions of our theoretical results are quite restrictive, the numerical results confirm that the reconstruction algorithm also works well if these assumptions are not fully met.

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Data availability statement

The data that support the findings of this study are openly available at the following URL/DOI: <https://gitlab.kit.edu/kir/ianm/ag-ip/software/2025-rg-ls-ff-op-splitting-low-rank>.

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